Analytical solution of heat transfer in a cone made of functionally graded material

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Abstract

In the current study, the problem of two-dimensional steady-state heat conduction in a truncated hollow cone made of functionally graded materials is referred and an exact analytical solution is presented. In the present study, the properties of a material are modified in accordance with a power function. The thermal boundary conditions are also assumed to be non-homogeneous. The separation of variable (SOV) method is implemented to acquire the exact steady-state temperature distribution. The obtained solution is adequately verified using numerical data. To further demonstrate the ability of the solution, an illustrative case that is exposed to a combination of boundary conditions is studied. In particular, the influences of effective parameters on the temperature distribution are investigated for the current geometry. The outcome of this study would be helpful to shed light on the process of designing and optimizing relatively complex geometries. Also, considering the analyticity of the present solution, the results of this study can be useful for a better understanding of the heat transfer mechanisms of functionally graded materials. In the present case, increasing the amount of m and κ, the thermal conductivity increased by about 8 and 2 percent respectively, which would increase the distribution of cone temperature.

Keywords: Functionally graded materials, Heat Transfer, Exact Solution, Cone, Separation of variable method

1- Introduction

Functionally graded material (FGM) has grown in popularity due to their widespread applications in different fields. The foremost feature of such material is their high thermal-resistance [1]. Consequently, it is of great importance to study them in terms of thermal behavior to be able to hire them for practical usages. There are different methods that have been employed to study the temperature distribution (*θ*) of FGMs. Out many, analytical solutions have shown robust and solid outcomes regarding these problems. So far, rather simple geometries like spherical and cylindrical shells have been studied widely [2, 3], while more complicated ones like truncated cones have not been studied. Thus, in this study, the problem of two-dimensional (2D) heat conduction on a truncated conical shell which is made of FGM whose material properties vary in accordance with a power function is analytically (using SOV) investigated. After validating the obtained exact solution, some parametric studies using a practical example are considered.

2- Methodology

In this section, the governing equations along with employed boundary conditions are noted. The geometry of the cone and imposed boundary conditions are illustrated in Fig. 1. It must be noted that the internal wall of the cone is considered to be insulated and there is convection current through its thickness. Due to the fact, that FGMs' features are required to vary in different directions, its density, for example, has to change according to a function. Mostly, a power-law function is hired to fulfill such an important matter. Heat conductivity (*k*) is defined through Eqs. 1-3 with k_1، k_2 and m being constant values. Moreover, since 2D heat transfer (*y* and *φ*) is considered to be orthotropic, there are only the values of k_y and k_φ [4].

$$k_y = k_1 y^{-m} \tag{1}$$

$$k_\varphi = k_2 \tag{2}$$

$$\kappa = \frac{k_1}{k_2} \tag{3}$$

The steady-state equation of energy after some simplifications and using the assumed heat conductivity distribution along with the relation of $\theta(y,\varphi) = T(y,\varphi) - T_{r,\infty}$ can be written as in Eq. 4.

$$\frac{\partial}{\partial y}\left(k_y y \frac{\partial \theta}{\partial y}\right)+\frac{1}{y \sin^2\gamma}\frac{\partial}{\partial \varphi}\left(k_\varphi \frac{\partial \theta}{\partial \varphi}\right)-\frac{h_0}{\epsilon y}\theta = 0 \qquad (4)$$

The considered boundary conditions with regard to the stated notation in Fig. 1are as

$$\theta(a,\varphi) = 245\,\mathrm{K} \qquad (5)$$

$$h\theta(b,\varphi)+k\frac{\partial \theta(b,\varphi)}{\partial y} = q_0\left(0.2+2\cos\left(\varphi+\frac{\pi}{6}\right)\right)+h\theta_{y,\infty} \qquad (6)$$

q_0 is heat flux. The exact analytical solution can be achieved as

$$\kappa y^{-m}\left(y\frac{\partial^2\theta(y,\varphi)}{\partial y^2}+\frac{\partial \theta(y,\varphi)}{\partial y}(-m+1)\right) -\frac{h_0}{k_2\epsilon y}\theta(y,\varphi)+\frac{1}{y\sin^2\gamma}\frac{\partial^2\theta(y,\varphi)}{\partial \varphi^2} = 0 \qquad (7)$$

here h is the coefficient of convection.

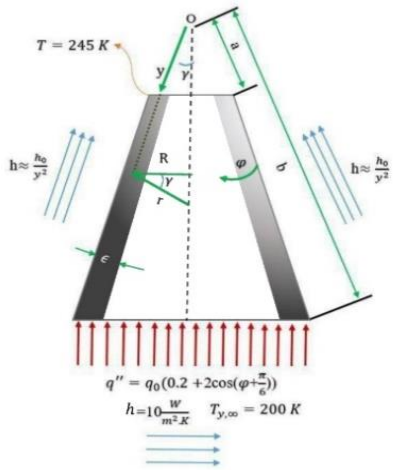

Fig. 1. Problem geometry and boundary conditions imposed on cones

Finally, heat distribution is as

$$\theta(y,\varphi) = y^{\frac{m}{2}}\left(A_0 I_1\left(\eta_0 y^{\frac{m}{2}}\right)+B_0 K_1\left(\eta_0 y^{\frac{m}{2}}\right)\right)+\sum_{n=1}^{\infty} y^{\frac{m}{2}}\left(A_n I_1\left(\eta y^{\frac{m}{2}}\right)+B_n K_1\left(\eta y^{\frac{m}{2}}\right)\right)\cos(\lambda_n\varphi)+\sum_{n=1}^{\infty} y^{\frac{m}{2}}\left(C_n I_1\left(\eta y^{\frac{m}{2}}\right)+D_n K_1\left(\eta y^{\frac{m}{2}}\right)\right)\sin(\lambda_n\varphi) \qquad (8)$$

The values of η,η_0 and eigenvalues can be determined as

$$\eta = -\frac{2}{m}\sqrt{\frac{h_0\sin^2\gamma+\lambda_n^{\,2}k_2\epsilon}{k_1\epsilon\sin^2\gamma}},\eta_0 = -\frac{2}{m}\sqrt{\frac{h_0}{k_1\epsilon}},\ \lambda_n = n \qquad n = 0,1,2,\ldots \qquad (9)$$

The constant coefficients are as follows:

$$A_0 = -\frac{245\mathcal{B}_n - K_1\left(\eta_0 a^{\frac{m}{2}}\right)\int_0^{2\pi}\left(q_0\left(0.2+2\cos\left(\varphi+\frac{\pi}{6}\right)\right)+h\theta_{y,\infty}\right)d\varphi}{\mathcal{A}_n K_1\left(\eta_0 a^{\frac{m}{2}}\right)-\mathcal{B}_n I_1\left(\eta_0 a^{\frac{m}{2}}\right)} \qquad (10)$$

$$B_0 = \frac{245\mathcal{A}_n - I_1\left(\eta_0 a^{\frac{m}{2}}\right)\int_0^{2\pi}\left(q_0\left(0.2+2\cos\left(\varphi+\frac{\pi}{6}\right)\right)+h\theta_{y,\infty}\right)d\varphi}{\mathcal{A}_n K_1\left(\eta_0 a^{\frac{m}{2}}\right)-\mathcal{B}_n I_1\left(\eta_0 a^{\frac{m}{2}}\right)} \qquad (11)$$

$$A_n = \frac{-\mathcal{Y}_n\frac{245}{\pi}\int_0^{2\pi}\cos(\lambda_n\varphi)d\varphi-\frac{1}{\pi}\int_0^{2\pi}\left(q_0\left(0.2+2\cos\left(\varphi+\frac{\pi}{6}\right)\right)+h\theta_{y,\infty}\right)\cos(\lambda_n\varphi)d\varphi\left(a^{\frac{m}{2}}K_1\left(\eta a^{\frac{m}{2}}\right)\right)}{\mathcal{X}_n a^{\frac{m}{2}}K_1\left(\eta a^{\frac{m}{2}}\right)-\mathcal{Y}_n a^{\frac{m}{2}}I_1\left(\eta a^{\frac{m}{2}}\right)} \qquad (12)$$

$$B_n = \frac{\mathcal{X}_n\frac{245}{\pi}\int_0^{2\pi}\cos(\lambda_n\varphi)d\varphi-\frac{1}{\pi}\int_0^{2\pi}\left(q_0\left(0.2+2\cos\left(\varphi+\frac{\pi}{6}\right)\right)+h\theta_{y,\infty}\right)\cos(\lambda_n\varphi)d\varphi\left(a^{\frac{m}{2}}I_1\left(\eta a^{\frac{m}{2}}\right)\right)}{\mathcal{X}_n a^{\frac{m}{2}}K_1\left(\eta a^{\frac{m}{2}}\right)-\mathcal{Y}_n a^{\frac{m}{2}}I_1\left(\eta a^{\frac{m}{2}}\right)} \qquad (13)$$

$$C_n = -\frac{\mathcal{Y}_n\frac{245}{\pi}\int_0^{2\pi}\cos(\lambda_n\varphi)d\varphi-\frac{1}{\pi}\int_0^{2\pi}\left(q_0\left(0.2+2\cos\left(\varphi+\frac{\pi}{6}\right)\right)+h\theta_{y,\infty}\right)\cos(\lambda_n\varphi)d\varphi\left(a^{\frac{m}{2}}K_1\left(\eta a^{\frac{m}{2}}\right)\right)}{\mathcal{X}_n a^{\frac{m}{2}}K_1\left(\eta a^{\frac{m}{2}}\right)-\mathcal{Y}_n a^{\frac{m}{2}}I_1\left(\eta a^{\frac{m}{2}}\right)} \qquad (14)$$

$$D_n = \frac{\mathcal{X}_n \frac{245}{\pi}\int_0^{2\pi}\cos(\lambda_n\varphi)\,d\varphi - \frac{1}{\pi}\int_0^{2\pi}\left(q_0\left(0.2+2\cos\left(\varphi+\frac{\pi}{6}\right)\right)+h\theta_{y,\infty}\right)\cos(\lambda_n\varphi)\,d\varphi\left(a^{\frac{m}{2}}I_1\left(\eta a^{\frac{m}{2}}\right)\right)}{\mathcal{X}_n a^{\frac{m}{2}}K_1\left(\eta a^{\frac{m}{2}}\right)-\mathcal{Y}_n a^{\frac{m}{2}}I_1\left(\eta a^{\frac{m}{2}}\right)} \quad (15)$$

Also, $\mathcal{A}_n$, $\mathcal{B}_n$, $\mathcal{X}_n$, and $\mathcal{Y}_n$ are as

$$\mathcal{A}_n = hb^{\frac{m}{2}}I_1\left(\eta_0 b^{\frac{m}{2}}\right)+k\frac{m\eta_0 b^{m-1}}{2}I_0\left(\eta_0 b^{\frac{m}{2}}\right),\ \mathcal{B}_n = hb^{\frac{m}{2}}K_1\left(\eta_0 b^{\frac{m}{2}}\right)+k\frac{m\eta_0 b^{m-1}}{2}K_0\left(\eta_0 b^{\frac{m}{2}}\right) \quad (16)$$

$$\mathcal{X}_n = hb^{\frac{m}{2}}I_1\left(\eta b^{\frac{m}{2}}\right)+k\frac{m\eta b^{m-1}}{2}I_0\left(\eta b^{\frac{m}{2}}\right),\ \mathcal{Y}_n = hb^{\frac{m}{2}}K_1\left(\eta b^{\frac{m}{2}}\right)+k\frac{m\eta b^{m-1}}{2}K_0\left(\eta b^{\frac{m}{2}}\right) \quad (17)$$

3- Results and Discussion

In this section, a practical test example is considered to do some parametric studies and further verify the obtained exact solution. The solution is verified against the result found via the finite element method (FEM). As can be seen from Fig. 2 there is a good match between the results. Figure 3 illustrates the distribution of temperature in both y and φ directions.

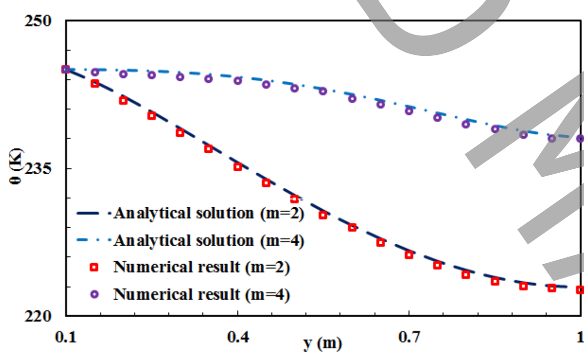

Fig. 2. Comparison of analytical solution and numerical results at $q_0 = 1357\ W/m^2$, $h_0 = 2\ W/K$ and $\kappa = 1$ for different values of m

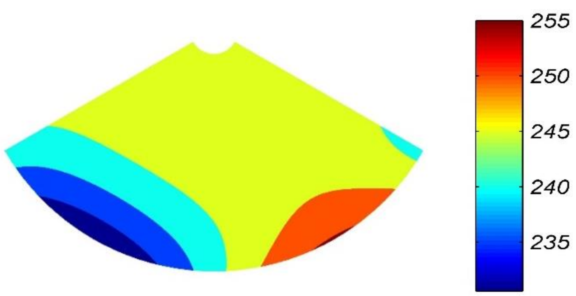

Fig. 3. Contours of temperature distribution in y and φ directions at $q_0 = 1357\ \mathrm{W/m^2}$, $\kappa = 1$, $h_0 = 2\ \mathrm{W/K}$ and $m = 1$

4- Conclusion

In the current paper, 2D temperature distribution in a truncated cone made of FGM is analytically studied. The separation of variable method is used for obtaining the exact solution. After the solution being validated against the numerical data, some parametric studies are done to shed light on the thermal behavior of FGM cones under asymmetric boundary condition. It is found out that $\boldsymbol{m}$ and $\boldsymbol{\kappa}$ are two important parameters in this matter and have strong effects on the shell's behavior. Increasing these two, would enhance heat conductivity.

The presented solution can be used for other similar problems with small modifications. Further, it would help to have a better understanding of the rather complicated geometries.

References

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