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Mixed Finite Element Formulation for 2D Problems Analysis Based on Analytical Solutions of Deferential Equation

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ABSTRACT: In this paper, a high-order eight-node element based on the analytical response of the governing differential equation is proposed for the analysis of plane structures. The formulation of the proposed element is based on the Hellinger-Reisner mixed functional and the analytical response of the compatibility equation governing plane problems. It is worth noting that in order to formulate finite elements with the Hellinger-Reisner functional, two independent stress and displacement fields are required. For this purpose, Airy stress functions are first made available by the analytical solution of the compatibility equation. By utilizing these stress functions, the stress field within the element is obtained. Also, the quadratic displacement field of the isoparametric element is used for intra-element displacement. By applying the Hellinger-Reisner mixed functional and stationary of this functional relative to the independent stress and displacement fields, the stiffness matrix, and the element node force vector are made available. Finally, with various numerical tests, the accuracy and efficiency of the proposed element are evaluated. These tests prove the high accuracy of the proposed element in the analysis of plane structures.

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1- Introduction

Since the advent of the finite element method, many elements have been proposed to analyze plane problems. Among these, the four-node and eight-node elements are all-encompassing, but these elements do not receive an acceptable response in distorted meshes, and the response error increases as the mesh distortion increases [1]. Therefore, many attempts have been made to find other types of formulation that have high accuracy and low sensitivity to mesh distortion. In this regard, we can refer to hybrid and mixed formulations [2]. In these formulation constructions, the number of main fields is more than one field.

Mixed functionals are established by defining independent fields within the element. One of the most popular mixed functionals for finite element formulation is the Hellinger-Reisner functional. In this function, the stress and displacement fields within the element are defined as independent fields. Due to the definition of the independent displacement field, one of the most important advantages of this method of the formulation is the liberation from the condition of continuity at the element boundaries [3]. Several high-order elements of 8 nodes and 12 nodes have also been created using hypothetical test functions and hybrid stress functional [4-6].

In this paper, using the Hellinger-Reisner functional, the formulation of an eight-node plate element with 16 degrees of freedom is performed. In the proposed element, instead of using hypothetical functions, the stress field is made available by the analytical solution of the governing differential equation. In this method, by solving the governance compatibility equation, the analytical functions of Airy stress are available. Then, by applying these stress functions, the stress field within the element is obtained. Also, for the intra-element displacement field, the eight-node isoparametric element interpolation functions are used. To demonstrate the accuracy and efficiency of the proposed element, various benchmark tests will be analyzed, and the answers will be compared with the results of other researchers' eight-node elements.

2- Methodology

2- 1- Hellinger-Reisner functional

To formulate the proposed element, the stress and displacement fields within the element are written as follows:

$$\{\sigma\}=[P]\{\beta\} \tag{1}$$

$$\{u\}=[N]\{D\} \tag{2}$$

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In the present equations, $[P]$ and $[N]$ are the assumed stress and displacement functions within the element, respectively. Also, vectors $\{\beta\}$ and $\{D\}$ are unknown stress factors and nodal displacement vectors, respectively. By placing Eqs. (1) and (2) in the Hellinger-Reisner functional, the energy of an element of the plane problem is written as follows:

$$\Pi_{HR} = \begin{pmatrix} \{\beta\}^T [G]\{D\} \\ -\frac{1}{2}\{\beta\}^T [H]\{\beta\} - \{f\}^T \{D\} \end{pmatrix} \quad (3)$$

By minimizing Eq. (3) relative to Vectors $\{\beta\}$ and $\{D\}$, stiffness and mass matrices are obtained.

$$[K] = [G]^T [H]^{-1} [G] \quad (4)$$

$$[G] = \int_v [P]^T ([L][N]) dv \quad (5)$$

$$[H] = \int_v [P]^T [C][P] dv \quad (6)$$

$$[L] = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \quad (7)$$

$$[C] = \frac{1}{E'} \begin{bmatrix} 1 & -\nu' & 0 \\ -\nu' & 1 & 0 \\ 0 & 0 & 2(1+\nu') \end{bmatrix} \quad (8)$$

For plane stress problems $E' = E$ and $\nu' = \nu$, and for plane strain problems $\nu' = \nu/(1-\nu)$ and $E' = E/(1-\nu^2)$.
The introduction should show the background of the subject and main contributions. It is necessary to explain clearly the novelty and contribution of present work in the last paragraph of introduction.

2- 2- Proposed element formulation

Fig. 1 shows the eight-node proposed element. For the internal displacement field of the proposed element, the interpolation functions of the Q8 element are used.

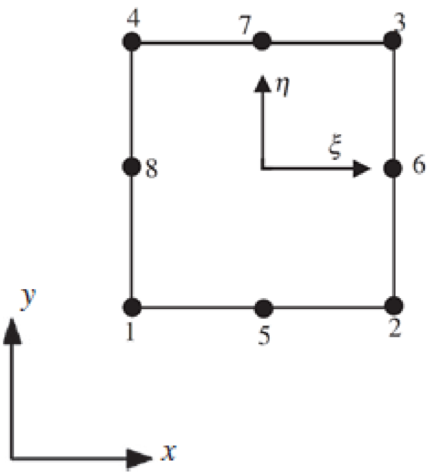

Fig. 1. The proposed element

$$\{u\} = \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = [N]\{D\}$$

$$N_j = \begin{pmatrix} \frac{1}{4}(1+\xi_j\xi)(1+\eta_j\eta) \\ (\xi_j\xi + \eta_j\eta - 1) \end{pmatrix} \quad for\ nodes\ j = 1,2,3,4 \quad (9)$$

$$N_j = \frac{1}{2}(1-\xi^2)(1+\eta_j\eta) \quad for\ nodes\ j = 5,7$$

$$N_j = \frac{1}{4}(1+\xi_j\xi)(1-\eta^2) \quad for\ nodes\ j = 6,8 \quad (10)$$

Also, the stress field of the component is calculated as the following matrix:

$$\sigma = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \sum_{i=1}^{n} \begin{Bmatrix} \frac{\partial^2 \phi_i}{\partial y^2} \\ \frac{\partial^2 \phi_i}{\partial x^2} \\ -\frac{\partial^2 \phi_i}{\partial x \partial y} \end{Bmatrix} \{\beta\} = [P]\{\beta\} \quad (11)$$

3- Results and Discussion

For proving the excellent efficiency of the proposed element, several different benchmarks are used. First, the beam with a length-to-thickness ratio of 4 is analyzed in a distorted four-element mesh under the effect of shear load with the proposed element. In the next test, MacNeal's beam with a length to thickness ratio of 30 under the effect of shear load and moment in different meshes is analyzed. The proposed element has high accuracy in both tests. Also, Cook's beam analysis with the proposed element proves the rapid rate of the new element. In order to evaluate the ability of the proposed element in the analysis of curved geometries, a curved beam was evaluated in both thick and thin modes under shear load. Finally, the sensitivity of the proposed element to mesh distortion was investigated. To do this, a mesh with two distorted elements was analyzed. The degree of mesh distortion depended on the distortion parameter. The proposed element, even in coarse meshes with high distortion, results in a high accuracy response.

4- Conclusion

In this study, using a mixed functional and analytical response of the differential equation of plane problems, an eight-node element with high accuracy was proposed to analyze the plane structures. To do this, the Hellinger-Reisner functional with independent stress and displacement fields was used. The intra-element stress field was obtained using the analytical response of the governing differential equation. Also, for the intra-element displacement field, the eight-node element interpolation functions were used. Then, by minimizing the functional equation with respect to the independent fields, the stiffness matrix and the vector node force vector became available. Finally, in order to measure and evaluate the accuracy of the proposed element, numerous numerical tests were performed. These tests showed the very high accuracy of the proposed element in the analysis of various plane structures and its low sensitivity to distortion of mesh.

References

[1] N.S. Lee, K.J. Bathe, Effects of element distortions on the performance of isoparametric elements, International Journal for numerical Methods in engineering, 36(20) (1993) 3553-3576.

[2] T.H. Pian, State-of-the-art development of hybrid/mixed finite element method, Finite elements in analysis and design, 21(1-2) (1995) 5-20.

[3] C. Felippa, Advanced finite element methods, Institute of Theoretical Physics, Faculty of Mathematics and Physics …, 2000.

[4] S. Cen, X.-R. Fu, M.-J. Zhou, 8-and 12-node plane hybrid stress-function elements immune to severely distorted mesh containing elements with concave shapes, Computer Methods in Applied Mechanics and Engineering, 200(29-32) (2011) 2321-2336.

[5] C. Wang, Y. Wang, C. Yang, X. Zhang, P. Hu, 8-node and 12-node plane elements based on assumed stress quasi-conforming method immune to distorted mesh, Engineering Computations, 34(8) (2017) 2731-2751.

[6] S. Cen, G.H. Zhou, X.R. Fu, A shape-free 8-node plane element unsymmetric analytical trial function method, International Journal for Numerical Methods in Engineering, 91(2) (2012) 158-185.

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