



## Boundary Conditions Identification in Problems of Hyper-Elastic Materials Deformation

M. Hajhashemkhani, M.R. Hematiyan\*

School of Mechanical Engineering, Shiraz University, Shiraz, Iran

**ABSTRACT:** In this study, by an inverse method, which uses the Tikhonov regularization method, traction boundary conditions on surface of a hyper-elastic material are determined. Displacements at several points on the surface of the body are measured and used to find the unknown stress parameters on a part of the problem boundary. The inverse analysis is carried out for Mooney-Rivlin and Ogden isotropic models. An example for identification of boundary conditions on a boundary part of a two dimensional domain with a relatively complicated geometry is presented to show the effectiveness of the proposed method. Effects of different parameters are studied in this example. The results for both hyper-elastic models show that the error of the solution decreases with increasing the number of measured data and decreasing the measurement error. Moreover, it is observed that the accuracy of the solution is decreased when the nonlinear behavior of the material is increased.

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### 1- Introduction

In some inverse problems, hyper-elastic material parameters are the unknowns of the problem. Czabanowski [1] by conducting some compression tests on elastomer samples and by using force-displacement data, obtained the unknown Mooney-Rivlin material parameters of the samples.

In some other inverse problems, boundary conditions are considered as the unknowns of the problem. Nakajima et al. [2] used ultrasonic measured displacement data and the boundary element method to obtain the unknown parameters of the boundary condition on the surface of the sample.

In this research, an inverse method for determining the boundary condition parameters of a hyper-elastic material with arbitrary shape is proposed. The Tikhonov regularization method is used in the inverse analysis.

### 2- Methodology, Discussion and Results

In hyper-elastic materials, stress-strain relationship is defined by strain energy density function in terms of the deformation gradient or strain tensors. Derivatives of the strain energy function with respect to strain give the stress components as follows:

$$S_{ij} = \frac{\partial \psi}{\partial \varepsilon_{ij}} \quad (1)$$

where  $S$  is the second piola-Kirchoff stress tensor and  $\varepsilon$  is the Lagrangian strain tensor defined as follows:

$$\varepsilon_{ij} = \frac{1}{2}(C_{ij} - \delta_{ij}) \quad (2)$$

$C$  is the right Cauchy-green deformation tensor and is defined as follows:

$$C = F^T F \quad (3)$$

where  $F$  is deformation gradient and is defined as follows:

$$F = \nabla u + I \quad (4)$$

where  $u$  is the displacement vector.

In general, strain energy density function can be expressed in terms of  $F$ . Therefore,  $\psi$  can be expressed in terms of  $C$ , i.e.,  $\psi = \psi(C)$  and the Cauchy stress tensor ( $\sigma$ ) is expressed as follows:

$$\sigma = 2J^{-1} F \frac{\partial \psi}{\partial C} F^T \quad (5)$$

For an isotropic material,  $\psi$  is dependent on  $C$  through its invariants. These invariants are defined as follows:

$$I_1 = tr C \quad (6)$$

$$I_2 = \frac{1}{2}[(tr C)^2 - (tr C^2)] \quad (7)$$

$$I_3 = \det C \quad (8)$$

For an isotropic material  $\psi$  is just dependent on  $I_1$ ,  $I_2$  and  $I_3$  and Equation 5 is expanded as follows:

$$\sigma = 2J^{-1} \left[ F \frac{\partial \psi}{\partial I_1} \frac{\partial I_1}{\partial C} F^T + F \frac{\partial \psi}{\partial I_2} \frac{\partial I_2}{\partial C} F^T + F \frac{\partial \psi}{\partial I_3} \frac{\partial I_3}{\partial C} F^T \right] \quad (9)$$

Corresponding author, E-mail: mhemat@shirazu.ac.ir

Strain energy density function for Mooney-Rivlin hyper-elastic model is defined as follows:

$$\psi = C_{10}(\bar{I}_1 - 3) + C_{01}(\bar{I}_2 - 3) + D_1(J-1)^2 \quad (10)$$

where,  $\bar{I}_1 = J^{2/3}I_1$  and  $\bar{I}_2 = J^{4/3}I_2$  and  $C_{10}$ ,  $C_{01}$  and  $D_1$  are material constants.

The strain energy density function for Ogden hyper-elastic model is defined as follows:

$$\psi = \sum_{i=1}^N \frac{2\mu_i}{\alpha_i^2} (\bar{\lambda}_1^{\alpha_i} + \bar{\lambda}_2^{\alpha_i} + \bar{\lambda}_3^{\alpha_i} - 3) + \sum_{i=1}^N \frac{1}{D_i} (J-1)^{2i} \quad (11)$$

where,  $\bar{\lambda}_1$ ,  $\bar{\lambda}_2$  and  $\bar{\lambda}_3$  are principle stretches and  $N$ ,  $\mu_i$ ,  $\alpha_i$  and  $D_i$  are material constants.

In the inverse problem to be analyzed here, the stress distribution in a part of the boundary is the unknown of the problem. The stress distribution is considered to be piecewise quadratic.

The vector of unknowns is expressed as follows:

$$\mathbf{m} = [m_1 \cdots m_k] \quad (12)$$

Further, the vector of measured displacements at sampling points and the vector of displacements obtained from solving the direct problem using initial guesses are named as  $\mathbf{y}$  and  $\mathbf{u}$ , respectively. Based on the Tikhonov regularization method the cost function of a minimization problem is defined as follows:

$$S = (\mathbf{y} - \mathbf{u})(\mathbf{y} - \mathbf{u})^T + \mu \mathbf{m}^T \mathbf{m} \quad (13)$$

where,  $\mu$  is the regularization parameter. By minimizing  $S$  with respect to  $\mathbf{m}$  the following relation is obtained:

$$\frac{\partial S}{\partial \mathbf{m}} = -2\mathbf{X}^T(\mathbf{y} - \mathbf{u}) + 2\mu \mathbf{m} = 0 \quad (14)$$

where,  $\mathbf{X}$  is the sensitivity matrix.

The unknown vector  $\mathbf{m}$  can be obtained by an iterative method. Suppose that  $\tilde{\mathbf{m}}$  and  $\tilde{\mathbf{u}}$  are the unknown vector and displacement vector at current step (iteration), respectively. The displacement vector at the next step can be updated as follows:

$$\mathbf{u} = \tilde{\mathbf{u}} + \mathbf{X}(\mathbf{m} - \tilde{\mathbf{m}}) \quad (15)$$

By substituting Equation (15) in (14) and after some mathematical manipulations, the following equation is obtained:

$$\mathbf{m} = [\mathbf{X}^T \mathbf{X} + \mu \mathbf{I}]^{-1} [\mathbf{X}^T(\mathbf{y} - \tilde{\mathbf{u}}) + \mathbf{X}^T \mathbf{X} \tilde{\mathbf{m}}] \quad (16)$$

Equation (16) is used in an iterative process.

The convergence rule for the iterative process is defined as follows:

$$\|\mathbf{m}^{k+1} - \mathbf{m}^k\| \leq e \quad (17)$$

In this research, in order to provide measurement data for an example, a direct problem with five known boundary condition parameters ( $\sigma_1$  to  $\sigma_5$ ) is solved. Some errors were added to the results obtained from the direct analysis to simulate the measurement errors. In the first case, the Mooney-Rivlin hyper-elastic model with material constants of 80Pa, 20Pa and 0 was employed. The loaded sample is shown in Figure 1.

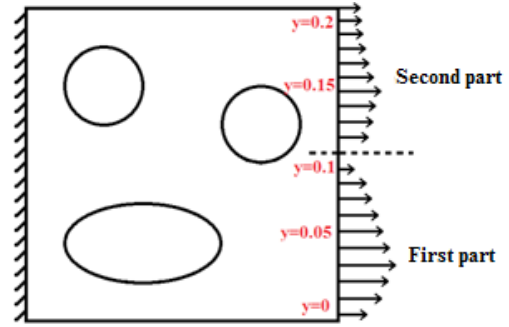


Figure 1. Loaded Sample

The effect of the location of measurement points on the results was studied first. It was concluded that, when the measurement points were located on the edge with the unknown boundary condition, the error in the obtained results was less. The effect of the number of measurement points on the results was also studied. It was found that by increasing the number of measurement points the error in results and the number of iterations decrease.

The effect of the magnitude of measurement errors on results was also studied. It was observed that an increase in measurement error increases the error in obtained constants and increases the number of iterations in the inverse analysis. By changing the material constants, the nonlinearity of the problem was increased and consequently the number of iterations and the error in the results increased too.

The analyses were performed on Ogden hyper-elastic model as well and the results were consistent with those obtained for Mooney-Rivlin hyper-elastic model.

### 3- Conclusions

In this research, an inverse problem for hyper-elastic materials with unknown boundary conditions was introduced. The unknown parameters were obtained through an inverse iterative analysis. It was observed that by increasing the number of measurement points the accuracy of the results is improved. The increase in the number of measurement points must provide useful information for the problem. In choosing the number and location of the measurement points one must note that, the quantity, which is being measured at the measurement points is sensitive enough with respect to the unknown parameters. Further, it was seen that increase in measurement error, increases the error in results enormously. If the nonlinearity of the problem is increased, even with small measurement errors, the results are obtained with a considerable error.

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