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Analytical Solution for Temperature, Stress and Displacement Fields for a Hollow Cylinder Subjected to Asymmetric and Time Dependent Heat Flux

A. Bidzard*, M. Mahzoon, M. Yaghoubi

Mechanical Engineering Department, Shiraz University, Shiraz, Iran

ABSTRACT: In present paper, a two dimensional analytical solution for temperature, stress and displacement fields in a hollow cylinder is developed. An asymmetric and time dependent heat flux is exposed on outer surface of the hollow cylinder. Moreover, the cylinder carries a fluid that transfers heat through convection on its inner surface. The separation of variable method is implemented to obtain temperature field. Also, stress distribution is taken by means of thermal stress function method. Afterwards, displacement components are obtained by means of stress-strain and strain-displacement relations. The cylinder is regarded as a model for the absorber tube of parabolic trough collector. Using the analytical solution together with the actual properties of the model in solar power plant in Shiraz city, results are presented for a period of twelve hours from 06:00 a.m. till 06:00 p.m. The analytical solution is employed to extract numerical results using MATLAB software package. The results are also validated with those given by FEM, conducted via ANSYS computer code. Finally, it is concluded that differences between the results of analytical solution and outputs of ANSYS are result of of infirmity of MATLAB software package in calculating the Kelvin functions with high accuracy.

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1- Introduction

Investigating temperature, stress and displacement fields, in wall of a hollow cylinder have always been of great interest, due to industrial applications. Ghosn and Sabbaghian [1] investigated quasi-static coupled problems of thermo-elasticity for cylindrical regions. They used Laplace transform in conjunction with calculus of residues and convolution theorem to present their solution. Goshima and Miyao [2] considered a long hollow circular cylinder exposed to transient internal heat generation and loss through convection. They used Laplace transform and Green function to study the problem.

In the present paper, initially, a two dimensional analytical solution for temperature distribution in a hollow cylinder by means of the separation of variable method is presented. As for the boundary condition, asymmetric and time dependent heat flux is exerted on the outer surface. Moreover, the cylinder carries a fluid that transfers heat through convection on its inner surface. Afterwards, thermal stress is taken by means of thermal stress function. Finally, displacement components are obtained by means of stress-strain and strain-displacement relations.

2- Mathematical Modelling

We can consider an infinite hollow cylinder of the inner and outer radius of $r_i=a$ and $r_o=b$, respectively. The two-dimensional heat diffusion equation in cylindrical coordinates is

$$\frac{1}{k}\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} \qquad (a\le r\le b, 0\le\theta\le 2\pi, t\ge 0) \tag{1}$$

Corresponding author, E-mail: adel.bidzard@yahoo.com

where, r is the radial distance, and k is the thermal diffusivity which is defined as

$$k=\frac{\lambda}{\rho c} \tag{2}$$

$$\tau=T-T_0 \tag{3}$$

where λ is thermal conductivity, ρ is the density, c is the specific heat and T_0 is initial temperature . The solution $T(r,\theta,t)$ must satisfy the boundary conditions given in Eqs. 4 and 5.

$$\lambda\frac{\partial \tau(a,\theta,t)}{\partial r}=h\tau(a,\theta,t) \qquad (0\le\theta\le 2\pi, t\ge 0) \tag{4}$$

$$\lambda\frac{\partial \tau(b,\theta,t)}{\partial r}=f(\theta)\cos(wt) \qquad (0\le\theta\le 2\pi, t\ge 0) \tag{5}$$

3- Temperature Distribution

We can construct a solution for Eq. 1 by the separation of variable method as

$$\tau(r,\theta,t)=R(r)\phi(\theta)M(t) \tag{6}$$

Substituting Eq. 6 into Eq. 1 and applying the boundary conditions. Following solution is derived.

$$\begin{aligned}\tau(r,\theta,t)=\sum_{n=0}^{\infty}[&C_n(ber_n(\sqrt{\tfrac{w}{k}}r)+ibei_n(\sqrt{\tfrac{w}{k}}r))\\&+D_n(Ker_n(\sqrt{\tfrac{w}{k}}r)+iKei_n(\sqrt{\tfrac{w}{k}}r))\times(A_n\cos n\theta+B_n\sin n\theta)e^{iwt}]\end{aligned} \tag{7}$$

$$\tau(r,\theta,t)=\sum_{n=0}^{\infty}\left(\frac{(z\cos wt+y\sin wt)M-(z\sin wt-y\cos wt)N}{-2\pi\lambda(z^2+y^2)}\right)\times\left(\int_0^{2\pi}f(s)\cos n(\theta-s)ds\right) \quad (8)$$

$$\begin{cases} z=[Gber_n'(\sqrt{\frac{w}{k}}b)+Ker_n'(\sqrt{\frac{w}{k}}b)-Hbei_n'(\sqrt{\frac{w}{k}}b)] \\ y=[Gbei_n'(\sqrt{\frac{w}{k}}b)+Kei_n'(\sqrt{\frac{w}{k}}b)+Hber_n'(\sqrt{\frac{w}{k}}b)] \\ G=\frac{pr+qs}{r^2+s^2};\quad H=\frac{qr-ps}{r^2+s^2} \\ p=-\gamma Ker_n'(\sqrt{\frac{w}{k}}a)+hKer_n(\sqrt{\frac{w}{k}}a) \\ q=-\gamma Kei_n'(\sqrt{\frac{w}{k}}a)+hKei_n(\sqrt{\frac{w}{k}}a) \\ r=\gamma ber_n'(\sqrt{\frac{w}{k}}a)-hber_n(\sqrt{\frac{w}{k}}a) \\ s=\gamma bei_n'(\sqrt{\frac{w}{k}}a)-hbei_n(\sqrt{\frac{w}{k}}a) \\ M=G\cdot ber_n(\sqrt{\frac{w}{k}}r)+Ker_n(\sqrt{\frac{w}{k}}r)-H\cdot bei_n(\sqrt{\frac{w}{k}}r) \\ N=H\cdot ber_n(\sqrt{\frac{w}{k}}r)+G\cdot bei_n(\sqrt{\frac{w}{k}}r)+kei_n(\sqrt{\frac{w}{k}}r) \end{cases} \quad (9)$$

4- Stress Distribution

Temperature distribution is given in following format

$$\tau(r,\theta,t)=e^{iwt}\left\{\sum_{n=0}^{\infty}F_n(r)\cos(n\theta)+\sum_{n=1}^{\infty}G_n(r)\sin(n\theta)\right\} \quad (10)$$

where

$$\begin{cases} F_n(r)=A_n Br_n(x)+B_n Kr_n(x) \\ G_n(r)=C_n Br_n(x)+D_n Kr_n(x) \\ Br_n(x)=ber_n(x)+ibei_n(x) \\ Kr_n(x)=\ker_n(x)+ikei_n(x) \\ x=\sqrt{\frac{w}{k}}r \end{cases} \quad (11)$$

For the plane strain problem, the compatibility equation can be written in terms of stress function

$$\nabla^4\chi+\frac{E\alpha}{1-\nu}\nabla^2\tau=0 \quad (12)$$

Where χ is thermal stress function notation. The stresses can now be determined in terms of stress function given by

$$\begin{cases} \sigma_r=\frac{1}{r}\frac{\partial\chi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2} \\ \sigma_\theta=\frac{\partial^2\chi}{\partial r^2} \\ \sigma_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta}\right) \end{cases} \quad (13)$$

If we suppose

$$\chi(r,\theta,t)=\left\{\sum_{n=0}^{\infty}f_n(r)\cos(n\theta)+\sum_{n=1}^{\infty}g_n(r)\sin(n\theta)\right\}e^{iwt} \quad (14)$$

Finally, we have for $n=1$

$$\begin{cases} \sigma_r=-\left\{2C_2r^{-3}-2C_3r-C_4r^{-1}+\frac{E\alpha r^{-3}}{1-\nu}\int_{r_i}^{r}F_1(s)s^2ds\right\}\cos\theta \\ -\left\{2D_2r^{-3}-2D_3r-D_4r^{-1}+\frac{E\alpha r^{-3}}{1-\nu}\int_{r_i}^{r}G_1(s)s^2ds\right\}\sin\theta \\ \sigma_\theta=\left\{2C_2r^{-3}+6C_3r+C_4r^{-1}+\frac{E\alpha}{1-\nu}\left[r^{-3}\int_{r_i}^{r}\left(F_1(s)s^2ds\right)-F_1(r)\right]\right\}\cos\theta \\ +\left\{2D_2r^{-3}+6D_3r+D_4r^{-1}+\frac{E\alpha}{1-\nu}\left[r^{-3}\int_{r_i}^{r}\left(G_1(s)s^2ds\right)-G_1(r)\right]\right\}\sin\theta \\ \sigma_{r\theta}=-\left\{2C_2r^{-3}-2C_3r-C_4r^{-1}+\frac{E\alpha r^{-3}}{1-\nu}\int_{r_i}^{r}F_1(s)s^2ds\right\}\sin\theta \\ +\left\{2D_2r^{-3}-2D_3r-D_4r^{-1}+\frac{E\alpha r^{-3}}{1-\nu}\int_{r_i}^{r}G_1(s)s^2ds\right\}\cos\theta \end{cases} \quad (15)$$

where α is the coefficient of thermal expansion, ν is Poisson's ratio, and E is the Young's modulus of elasticity.

5- Displacement Distribution

In linear thermo-elasticity we have

$$\begin{cases} \varepsilon_r=\frac{\partial u_r}{\partial r}=\frac{1+\nu}{E}\left[(1-\nu)\sigma_r-\nu\sigma_\theta\right]+(1+\nu)\alpha\tau \\ \varepsilon_\theta=\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}=\frac{1+\nu}{E}\left[(1-\nu)\sigma_\theta-\nu\sigma_r\right]+(1+\nu)\alpha\tau \\ \varepsilon_{r\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)=\frac{1+\nu}{E}\sigma_{r\theta} \end{cases} \quad (16)$$

Substituting for σ_r and σ_θ from Eq. 15 and τ from Eqs. 8 into Eq. (16) and subsequent integrating yields

$$\begin{aligned} u_r=&\cos\theta\left\{\frac{1+\nu}{E}\left[C_2r^{-2}+(1-4\nu)C_3r^2+(1-2\nu)C_4\ln r\right]\right. \\ &\left.+\frac{(1+\nu)\alpha}{2(1-\nu)}\left[r^{-2}\int_a^r F_1(s)s^2ds+\int_a^r F_1(s)ds\right]\right\} \\ +&\sin\theta\left\{\frac{1+\nu}{E}\left[D_2r^{-2}+(1-4\nu)D_3r^2+(1-2\nu)D_4\ln r\right]\right. \\ &\left.+\frac{(1+\nu)\alpha}{2(1-\nu)}\left[r^{-2}\int_a^r G_1(s)s^2ds+\int_a^r G_1(s)ds\right]\right\}+U(\theta) \end{aligned} \quad (17)$$

$$\begin{aligned} u_\theta=&\sin\theta\left\{\frac{1+\nu}{E}\left[C_2r^{-2}+(5-4\nu)C_3r^2+(1-2\nu)C_4(1-\ln r)\right]\right. \\ &\left.+\frac{(1+\nu)\alpha}{2(1-\nu)}\left[r^{-2}\int_a^r F_1(s)s^2ds-\int_a^r F_1(s)ds\right]\right\} \\ -&\cos\theta\left\{\frac{1+\nu}{E}\left[D_2r^{-2}+(5-4\nu)D_3r^2+(1-2\nu)D_4(1-\ln r)\right]\right. \\ &\left.+\frac{(1+\nu)\alpha}{2(1-\nu)}\left[r^{-2}\int_a^r G_1(s)s^2ds-\int_a^r G_1(s)ds\right]\right\}-\int U(\theta)d\theta+v(r) \end{aligned} \quad (18)$$

u_r and u_θ are radial and circumferential displacements and $V(r)$ and $U(\theta)$ represent rigid displacement and rigid rotation.

6- Results and Discussion

Comparison of temperature change in analytical solution and FEM for actual model of absorber tube in solar power plant in Shiraz city are shown in Figure. 1. The reason of differences is in infirmity of MATLAB software package in calculation of the Kelvin functions with high accuracy.

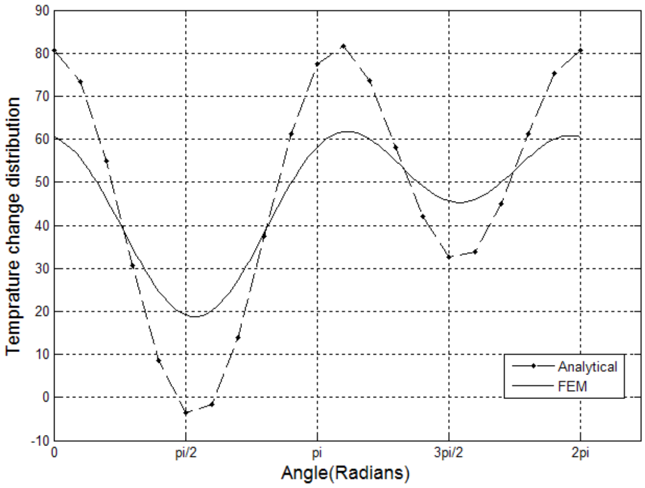

Figure 1. Comparison of temprature change distribution, analytical versus FEM

7- Conclusions

The separation of variable method is used to obtain temperature field. Also, stress distribution is taken by means of thermal stress function method. Afterwards, displacement components are obtained by means of stress-strain and strain-displacement relations. We conclude that differences between the results of analytical solution and output of ANSYS computer code are result of of infirmity of MATLAB software package in calculating the Kelvin functions precisely.

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