Mathematical Modelling and Resonance Analysis in Impact Oscillators

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ABSTRACT: A variety of mechanical phenomena can be simulated by modeling bilinear oscillators which have different stiffness in pressure and tension. In this paper, bilinear oscillators with unlimited stiffness in compression say impact oscillators, and the eigenset which is homogenous solution of equation without external load are investigated. The results show that this set and the corresponding subsets are stable with respect to variation in initial conditions. In addition, among all periodic collections of impact times which are proportional to the period of external load, only the eigenset can support resonance, especially the multi-harmonic resonance. The rest of the resonances should produce the non-periodic impact times. This phenomenon shows that the usual assumption that the times between impacts are proportional to the period of external load is not always confirmed. Furthermore it is shown that in half frequency of the main resonance (the first sub-harmonic resonance), the impact times are close to the eigenset and unlike linear increase of multi-harmonic resonances, the envelope of the oscillations increases as a square root of time.

1- Introduction

Bilinear oscillators which have different stiffness in compression and tension, simulate various engineering phenomena. One of the mechanisms of these oscillators is the presence of contact which resists against in one direction such as cutting tools [1]. Another specific example is the friction between a rock and drill-string set up [2]. What is important in these applications is the probability of resonance in systems which by reducing friction resulting from fluctuations can lead to instability [3]. Bilinear oscillators which have infinity stiffness in compression are called impact oscillators. The most significant feature of impact oscillator is the impact times (the times of passing through the origin). So far only known analytical solutions for bilinear oscillators are related to impact oscillators [4].

According to the above-mentioned researches, it is necessary to devote further analysis to study the analytical methods and closed form solutions for impact oscillators. Thus, in this paper, we investigate the influence of impact times on resonances of multi-harmonics and sub-harmonics in impact oscillators by considering the eigenset proportional to the period of the external harmonic load. Asymptotical behavior of the system is subjected to the sinusoidal and cosine half harmonic resonance is investigated. The results are obtained by semi-analytical method. To evaluate the results by numerical simulations, Runge-Kutta method is applied for various parameters.

2- Methodology

The dimensionless form of equation of bilinear oscillations is introduced by [3]:

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$$x'' + 2\alpha x' + \tilde{k}(x)x = f(t)$$

$$\tilde{k}(x) = \begin{cases} \omega_+^2 & , \quad x \geq 0 \\ \omega_-^2 & , \quad x < 0 \end{cases} \quad , \quad \omega_+ < \omega_- \tag{1}$$

where $x(t)$ is oscillator displacement and α is the damping coefficient of the system. The main feature of bilinear oscillators is that nonlinearity in system occurs when the oscillator passes the origin. The frequency of resonance is obtained by summing up the times the oscillator spent in the each of linear directions [2]:

$$\Omega_r = \frac{2\Omega_+\Omega_-}{\Omega_+ + \Omega_-} \quad , \quad \Omega_\pm^2 = \omega_\pm^2 - \alpha^2 \tag{2}$$

By considering $\Theta = (t_k)$ as an impact time, the solution of Eq. (1) is obtained as:

$$x_0(t) = V_0 \sin(t - \pi k) \quad , \quad \pi k \leq t \leq \pi(k+1) \tag{3}$$

where V_0 is the initial velocity of the oscillator in every interval. In sub-harmonic resonances, the impact times are obtained as:

$$\Theta_m = \{\pi s_k\} \subset \Theta_0 : s_0 = 0$$

$$s_{k+1} = \begin{cases} s_k + m & k = 2n \\ s_k + 1 & k = 2n+1 \end{cases} \tag{4}$$

Also, by considering a group of the following functions:

$$x_*(t) = A\sin t - \cos\frac{2t + \pi}{m+1} + \cos\frac{\pi}{m+1} \tag{5}$$

which all of the mentioned functions satisfy Eq. (1) with the external load as illustrated below:

$$f(t) = \frac{(3-m)(m+5)}{(m+1)^2} \cos \frac{2t+\pi}{m+1} + \cos \frac{\pi}{m+1} \qquad (6)$$

when m is odd, the initial velocity has a boundary for occurrence regular impact times which is presented in Table 1.

Table 1. Boundary values of initial velocities for

m	V_*(m)	m	V_*(m)
5	1.5	11	0.486
7	0.984	13	0.365
9	0.675	15	0.283

If one assumes that the set Θ of the impact times has period T which is the same as external load period, by considering Eq. (1), impact times should coincide with eigenset. By analyzing the equation as follows:

$$x'' + x = \sin nt \qquad , \qquad x > 0 \qquad (7)$$

Period of the external load is $2\pi/n$. When n is even, the period of eigenset is equal or multiple of $2\pi/n$. then resonance is possible and the solution is [1, 3]:

$$x_k(t) = A_k \sin t - \frac{1}{n^2-1} \sin nt \qquad (8)$$

$$x'(t) = A_k \cos t - \frac{1}{n^2-1} \cos nt$$

$$V_k = V_{k-1} + \frac{2n}{n^2-1} \qquad (9)$$

$$A_k = (-1)^k \left[V_k + \frac{n}{n^2-1} \right]$$

where V_k is the initial velocity in the beginning of each cycle. Since the frequency of Eq. (2) is 2, multi-harmonic resonance occurs and amplitude and impact velocity increase linearly. By considering the impact times in the sub-harmonic resonance:

$$x'' + x = \sin t \qquad , \qquad x > 0 \qquad (10)$$

the solution of above equation can be expressed as:

$$x_k(t) = A_k \sin t + B_k \cos t - \frac{t}{2} \cos t \qquad (11)$$

$$x'_k(t) = \left(A_k - \frac{1}{2} \right) \cos t - B_k \sin t + \frac{t}{2} \sin t \qquad (12)$$

By introducing $\xi_k = t_{k+1} - t_k$, which t_k and t_{k+1} are consecutive impact times, the roots of Eq. (12) can be written as:
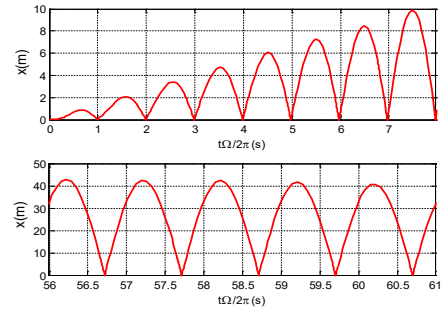
$$\tan \xi_k = \frac{\cos t_k}{\cos t_k + 2V_k} \xi_k \qquad (13)$$

The asymptotic solution of the above equation is determined by:

$$\frac{dV}{dt} = \frac{a}{\pi V} + 0(V^{-2}) \qquad (14)$$

3- Results and Discussion

Any minor deviation from the resonance frequency leads to produce the non-periodic impact times (see Fig. 1).

Fig. 1. Numerical solution of Eq. (1) and occurrence of non-periodic response of bilinear oscillator ($\omega_-^2/\omega_+^2=20$, $\alpha=0$, $\omega=1.01\Omega_r$)

According to Eq. (14), the asymptotic envelope of the velocity indicates that for the desirable parameters, the numerical simulation is accurate.
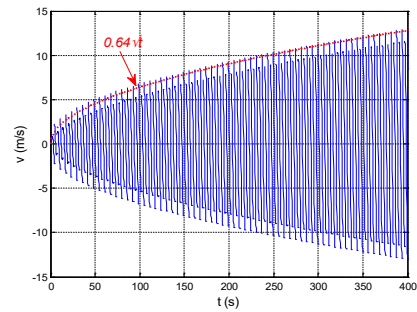
Fig. 2. The asymptotic envelope for velocity of sub harmonic resonance

According to Fig. 3, numerical simulation is comparing analytical prediction and excellent agreement was observed between methods. The results indicate that time between impact times are proportional to the period of the external load.
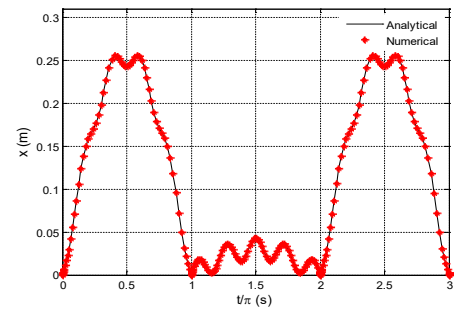
Fig. 3. Numerically and analytically calculated trajectory of impact oscillator ($n=9$, $V_0=0.139$)

4- Conclusion

In this paper, the influence of the impact times on resonance of impact oscillators which have infinity stiffness in compression was studied. Analytical expressions and Runge-Kutta method with adaptive step were used for the analysis. The results show that among all impact times with period proportional to the period of the external load, eigenset (the times passing the origin without external force) is the only case which can supports the resonances especially multi-harmonic resonances. Furthermore, the envelope of

the impact oscillations in the first sub-harmonic resonances increases as a square root of time opposite the linear increase of multi-harmonic resonances.

5- References

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