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A C[1] Finite Element Formulation for Mindlin-Reissner Microplate Model

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ABSTRACT: In this paper, a C[1] finite element (FE) formulation of Mindlin-Reissner microplate based on strain gradient elasticity theory is developed. The general form of the stiffness matrix and force vector of the microplate element is firstly extracted, and then specialized on a four-node quadrilateral element with 36 degrees of freedom. Deformation of rectangular microplates with simply-supported edges, clamped edges, and three edges simply-supported and the fourth edge free, and under uniform external pressure is then studied. For the case of microplate with simply-supported boundaries, comparison between the FE and the corresponding exact solution is made, which shows extremely close results. For the next two examples, a convergent solution by means of mesh refinement is obtained. Moreover, for the case of thin plates and for large values of the thickness-to-material length ratio, the results of gradient-based FE analysis are coincident with those of the Kirchhoff plate model based on classical elasticity. Numerical simulations show that the introduced element is able to capture the size effect phenomenon at micron scale. When the plate thickness is in the order of the material length parameter, the value of deflection is lower than that predicted by the models based on classical elasticity.

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1- Introduction

Various experimental results demonstrate that deformation of solids at micron scale is size-dependent [1-3]. Contrary to classical theory, in the strain gradient theory, originally developed by Mindlin [4], the strain energy function is a function of the strain as well as its derivative, and thus the stresses depend upon higher-order derivatives of the displacement field. It has been proven that strain gradient theory is effective in predicting the size-dependent elastic as well as plastic deformation of solids at micron scale [1-3, 5]. The simplest format of the theory, used in this work, was introduced by Aifantis [6] that contains a single material length scale parameter.

The objective of this work is to develop a four-node quadrilateral finite element (FE) for numerical analysis of gradient-elastic plate structures. A plate element, based on C[1]-continuous interpolation functions, for the numerical analysis of Reissner-Mindlin microplates is developed. The introduced element is based on Aifantis [6] form of strain gradient theory to capture size dependent behavior of plates at small scales.

2- Methodology

The strain energy density function U in the strain gradient theory of Aifantis [6] is given by

$$U = \frac{1}{2}\lambda\left(\varepsilon_{ii}\varepsilon_{jj} + l^2\xi_{ijj}\xi_{ikk}\right) + \mu\left(\varepsilon_{ij}\varepsilon_{ij} + l^2\xi_{ijk}\xi_{ijk}\right) \tag{1}$$

where ε_{ij} and ξ_{ijk} are the components of strain and strain gradient tensors, respectively. Additionally, l is the material length scale parameter in this theory. The stress and double stress tensors are calculated via $\sigma_{ij} = \partial U / \partial \varepsilon_{ij}$ and $\tau_{ijk} = \partial U / \partial \xi_{ijk}$, respectively.

Next, a microplate with constant thickness and material properties is considered. The displacement field corresponding to the flexural deformation of Mindlin-Reissner plate model is described by [7, 8]

$$u_\alpha = z\,\psi_\alpha, \quad u_3 = w \tag{2}$$

where $\psi_\alpha (\alpha = 1,2)$ are the rotation angles and w is the transverse deflection field. The components of strain and strain gradient tensors are calculated based on Eq. (2). Then, the stress and double stress components can be calculated. The classical as well as non-classical resultants of the microplate are defined by

$$\begin{Bmatrix} M_{\alpha\beta} \\ Q_\alpha \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} z\,\sigma_{\alpha\beta} \\ \sigma_{\alpha 3} \end{Bmatrix} dz \tag{3}$$

$$\begin{Bmatrix} G_{\alpha\beta} \\ B_{\alpha\beta} \\ H_{\phi\alpha\beta} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \tau_{3\alpha\beta} \\ \tau_{\alpha\beta 3} \\ z\,\tau_{\phi\alpha\beta} \end{Bmatrix} dz \tag{4}$$

For FE formulation using virtual work principle, the variation of strain energy density is decomposed into two parts as $\delta U = \delta U^{(1)} + \delta U^{(2)}$, with

$$\{\delta U^{(1)}, \delta U^{(2)}\} = \int_V \{\sigma_{ij}\,\delta\varepsilon_{ij}, \tau_{ijk}\,\delta\xi_{ijk}\} dV \tag{5}$$

Next, the vectors $\mathbf{F}$ and $\boldsymbol{\omega}$ are defined by

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$$\mathbf{F} = \{M_1, M_2, M_{12}, Q_1, Q_2\}^T \tag{6}$$

$$\omega = \{\kappa_{11}, \kappa_{22}, 2\kappa_{12}, \gamma_{13}, \gamma_{23}\}^T \tag{7}$$

where $\kappa_{\alpha\beta} = \left(\psi_{\alpha,\beta} + \psi_{\beta,\alpha}\right)/2$ and $\gamma_{\alpha 3} = \psi_\alpha + w_{,\alpha}$. It is noted that the relation $\mathbf{F} = \tilde{\mathbf{D}}\omega$ holds, where $\tilde{\mathbf{D}}$ is the classical plate stiffness matrix. Accordingly, the expression for $\delta U^{(1)}$ takes the following simple form:

$$\delta U^{(1)} = \int_A \delta\dot{\mathbf{u}}\,.\mathbf{F}\,dA = \int_A \delta\dot{\mathbf{u}}\,.\left(\tilde{\mathbf{D}}\dot{\mathbf{u}}\right)dA \tag{8}$$

Now, a microplate element with n arbitrary nodes is considered. The rotation and deflection fields are interpolated via the relation

$$\begin{Bmatrix} \psi_1 \\ \psi_2 \\ w \end{Bmatrix} = \begin{Bmatrix} \mathbf{Nd}_1 \\ \mathbf{Nd}_2 \\ \mathbf{Nd}_3 \end{Bmatrix} = \sum_{I=1}^{3n} N_I \begin{Bmatrix} d_{1I} \\ d_{2I} \\ d_{3I} \end{Bmatrix} \tag{9}$$

Where N_I are the interpolation functions. By Eqs. (7) and (9), the generalized strain vector $\boldsymbol{\omega}$ is discretized by the relation $\boldsymbol{\omega} = \mathbf{B}\,\mathbf{d}^e$, where $\mathbf{B}$ is a generalized strain-displacement matrix. Accordingly, Eq. (8) is rewritten as

$$\delta U^{(1)} = \delta\mathbf{d}^e.(\mathbf{K}_1^e\mathbf{d}^e) \tag{10}$$

where $\mathbf{K}_1^e$ is the classical part of the element stiffness matrix given by

$$\mathbf{K}_1^e = \int_A \mathbf{B}^T\tilde{\mathbf{D}}\mathbf{B}\,dA \tag{11}$$

Similar to what was done for the classical part, the gradient-based vectors $\mathbf{F}^*$ and $\boldsymbol{\omega}^*$ are defined as follows:

$$\mathbf{F}^* = \{H_{111}, H_{222}, H_{122}, H_{211}, H_{112}, H_{212}, G_{11}, G_{22}, G_{12}, B_{11}, B_{22}, B_{12}, B_{21}\}^T \tag{12}$$

$$\dot{\mathbf{u}}^* = \{\xi^*_{111}, \xi^*_{222}, \xi^*_{122}, \xi^*_{211}, 2\xi^*_{112}, 2\xi^*_{212}, \xi_{113}, \xi_{223}, 2\xi_{123}, 2\xi_{311}, 2\xi_{322}, 2\xi_{312}, 2\xi_{321}\}^T \tag{13}$$

where the matrix relation $\mathbf{F}^* = \tilde{\mathbf{D}}^*\omega^*$ holds. Using Eqs. (9) and (13), the discretized form of $\boldsymbol{\omega}^*$ is given by $\boldsymbol{\omega}^* = \mathbf{B}^*\mathbf{d}^e$, where $\mathbf{B}^*$ is the corresponding generalized strain gradient-displacement matrix. This together with Eqs. (12) and (13) yields the following expression for $\delta U^{(2)}$:

$$\delta U^{(2)} = \delta\mathbf{d}^e.(\mathbf{K}_2^e\mathbf{d}^e) \tag{14}$$

where $\mathbf{K}_2^e$ is the gradient part of the element stiffness matrix given by

$$\mathbf{K}_2^e = \int_A \mathbf{B}^{*T}\tilde{\mathbf{D}}^*\mathbf{B}^*\,dA \tag{15}$$

Finally, the element stiffness matrix of the strain gradient Mindlin-Reissner microplate model is given by the relation $\mathbf{K}^e = \mathbf{K}_1^e + \mathbf{K}_2^e$.
The element force vector depends on the applied external node. For instance, if p is the resultant external pressure on the element in $+z$ direction, and q_α are distributed moments per unit area around x_α, the element force vector is given by

$$\mathbf{f}^e = \int_A \{q_1\mathbf{N}, q_2\mathbf{N}, p\mathbf{N}\}^T\,dA \tag{16}$$

3- Results and Discussion

A rectangular plate of dimensions $L_1 \times L_2 \times h$, with L_α as the length of the plate along the x_α-coordinate, is considered. In the first example, a rectangular plate with simply-supported boundaries is considered. In the second one, the plate edges are assumed to be clamped. In the third examples, three edges are simply-supported and the forth one is free. To have non-dimensional results, the following non-dimensional parameters, similar to those defined in Ref. [9], are introduced:

$$X = \frac{h}{l},\quad Y = \frac{L_1}{h},\quad Z = \frac{L_1}{L_2},\quad W = \frac{w}{w^C} \tag{17}$$

where w^C is the maximum plate deflection based on the classical Kirchhoff plate model (e.g., Ref. [10]).
In the first two examples, due to symmetry, only one-quarter of the geometry is discretized by $n\times n$ meshes of the newly introduced element. Convergence analysis shows that, in both examples, a 16×16 elements for mesh is sufficient to have convergent results for all values of the non-dimensional parameters *X*, *Y* and *Z*.
Now, the first example as a square plate with simply-supported

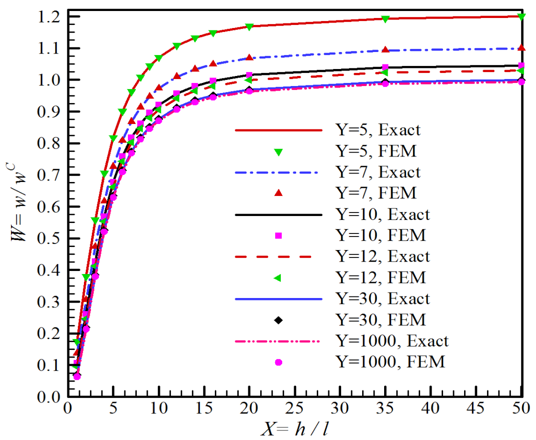

Fig. 1. Variation of the non-dimensional deflection *W* versus *X* and for various values of *Y* in a simply-supported square microplate

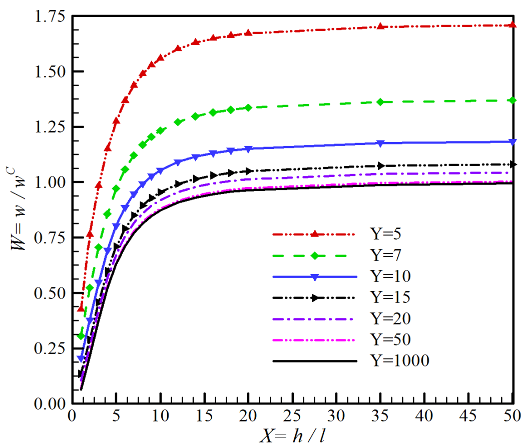

Fig. 2. Variation of the non-dimensional deflection *W* versus *X* and for various values of *Y* in a square microplate with clamped boundaries

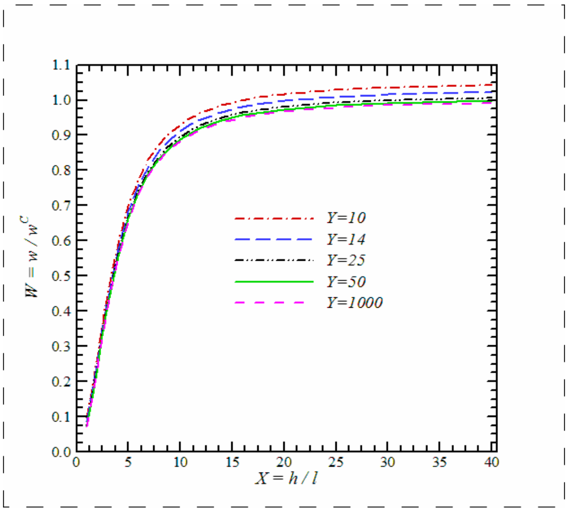

Fig. 3. Variation of the non-dimensional deflection *W* versus *X* and for various values of *Y* in a square microplate with three edges simply-supported and the fourth edges free

edges is considered. Variations of the nondimensional center deflection W versus the ratio of thickness-to-material length scale parameter X, and for several values of the width-to-thickness ratio Y, have been illustrated in Fig. 1. As can be seen from the figure, the curves generated by the present finite element formulation and those based on the exact solution calculated based on the formulation in Ref. [9] are indistinguishable.

In the next example, a square plate with clamped boundaries under uniform pressure is considered. The nondimensional center deflection W versus variations of the X parameter and for several values of the width-to-thickness ratio Y has been illustrated in Fig. 2. By increasing the thickness, the nondimensional deflection W approaches to that predicted by the Mindlin-Reissner plate model based on classical elasticity theory. In the third example, a square plate with three edges simply-supported and the fourth edge free is considered. Convergence study reveals that a 16×32 elements for mesh is sufficient to have convergent results. The nondimensional center deflection W versus variations of the X parameter and for several values of the width-to-thickness ratio Y has been depicted in Fig. 3. Size effect is observed for $X < 30$.

4- Conclusions

In this work, a C^1 four-node quadrilateral microplate element for the analysis of Mindlin-Reissner microplates was developed. The formulation was based on the Aifantis form [6] of strain gradient elasticity theory to capture size effect phenomenon. By solving three examples, the capability and efficiency of the new element for a wide range of geometric parameters was investigated. It was shown that the new element can successfully capture the so-called size effect. Moreover, it was observed that the element behaves very well in moderately thick as well as very thin plates. Additionally, it was shown that the new element can regenerate the results of the Mindlin-Reissner plate model in the context of the classical continuum theory when the ratio of thickness-to-material length parameter, namely $X = h/l$, takes sufficiently large values.

References

[1] N.A. Fleck, G.M. Muller, M.F. Ashby, J.W. Hutchinson, Strain gradient plasticity: theory and experiment, *Acta Metallurgica et Materialia*, 42(2) (1994) 475–487.

[2] W.D. Nix, H.Gao, Indentation size effects in crystalline materials: a law for strain gradient plasticity, *Journal of the Mechanics and Physics of Solids*, 46(3) (1998) 411-425.

[3] D.C.C. Lam, F. Yang, A.C.M. Chong, P. Tong, Experiments and theory in strain gradient elasticity, *Journal of the Mechanics and Physics of Solids,* 51(8) (2003) 1477-1508.

[4] R.D. Mindlin, Micro-structure in linear elasticity, *Archive for Rational Mechanics and Analysis*, 16(1) (1964) 51-78.

[5] N.A. Fleck, J.W. Hutchinson, A reformulation of strain gradient plasticity, *Journal of the Mechanics and Physics of Solids*, 49(10) (2001) 2245–2271.

[6] E.C. Aifantis, On the role of gradients in the localization of deformation and fracture, *International Journal of Engineering Sciences*, 30(10) (1992) 1279-1299.

[7] I. Katili, A new discrete Kirchhoff-Mindlin element based on Mindlin-Reissner plate theory and assumed shear strain fields part II: an extended DKQ element for thick-plate bending analysis, *International Journal for Numerical Methods in Engineering*, 36 (11) (1993) 1885-1908.

[8] J. Dolbow, N. Moes, T. Belytschko, Modeling fracture in Mindlin-Reissner plates with the extended finite element method, *International Journal of Solids and Structures,* 37(48-50) (2000) 7161-7183.

[9] S. Ramezani, A shear deformation micro-plate model based on the most general form of strain gradient elasticity, *International Journal of Mechanical Sciences,* 57(1) (2012) 34-42.

[10] S. P. Timoshenko, S. Woinowsky-Kreiger, *Theory of plates and shells,* McGraw-Hill, 1959

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