Amirkabir Journal of Mechanical Engineering

Amirkabir J. Mech. Eng., 50(5) (2019) 353-356
DOI: 10.22060/mej.2018.14248.5823

A Semi-Analytical Method Based on the Mixed Formulation for the Elastic Analysis of a Crack in an Anisotropic Homogeneous Medium

S. Qasemi, M. T. Kamali*, B. Shokrolahi-Zadeh

Civil Engineering Department, University of Hormozgan, Bandar Abbas, Iran

ABSTRACT: In this study, a semi-analytical method based on the Reisner's mixed formulation is presented for the elastic analysis of anisotropic homogeneous solids with an edge or interior crack. In this method, the displacement and the stress fields are represented as the sum of a known function and a finite series of functions with unknown coefficients. The functions are constructed in such a way that the displacement discontinuity across the crack faces and the exact singular behavior of the stress field at the crack tip are captured, moreover, all essential and natural homogeneous and inhomogeneous boundary conditions are satisfied exactly. The equilibrium and the compatibility equations are also applied with the desired accuracy using the Reissner's variational principle. Solution of the variational equation leads to a set of linear algebraic equations in terms of the unknown coefficients. After computing of the unknown coefficients, the displacement and the stress fields are obtained and subsequently, the stress intensity factors are calculated. The results show that the computing of stress intensity factors has high convergence rate and the results of the proposed approach are in good agreement with those of the analytical solutions reported in the literature.

Review History:

Received: 2018-03-27
Revised: 2018-05-16
Accepted: 2018-05-25
Available Online: 2018-07-03

Keywords:

Stress intensity factor
Mixed formulation
Anisotropic elastic material
Reissner's variational principle

1. Introduction

Due to the singularity of the stress field at the crack-tip and the discontinuity of displacement field across the crack faces, the classical finite element method has some inconveniences in solving the crack problems. To overcome these inconveniences, some new numerical methods such as the Extended Finite Element Method (XFEM) and meshless methods have been employed. Using XFEM, analysis of a crack in an orthotropic material is investigated by Motamedi and Mohammadi [1]. Fleming et al. [2] have considered the crack problem using the enriched element free Galerkin method for crack tip fields. Kamali and Shodja [3] present a meshless semi-analytical method based on the Ritz method for the determination of the elastic fields within a cracked anisotropic homogeneous elastic solid. In the present study, extending of the work of Kamali and Shodja [3], a mixed formulation based on the Reisner's variational principle for elastic analysis of an anisotropic homogeneous medium with a crack is presented. In this paper, the stress field, as well as the displacement field, is expressed as a sum of a function and a finite series of functions with unknown coefficients. To examine the efficacy of the presented methodology, a cantilevered plate with an edge crack under in plane mixed mode loading is considered.

2. Problem Statement and Formulation

Consider a two-dimensional elastic homogeneous medium including an interior crack with length $2a$, as shown in Fig. 1. The origin of the Cartesian coordinate system is located in the middle of the crack and the x_1 axis is taken along the crack line. Two polar coordinates (r_1,θ_1) and (r_2,θ_2) are set at the crack-tips. The component of displacement and stress fields are given as:

$$\begin{aligned} u_i &= F_i(x_1,x_2) + B_i(x_1,x_2) \\ &\times\{\sqrt{\frac{r_1 r_2}{a^2}}[\sum_{m=0}^{n_r}\sum_{n=0}^{n_\theta}\left(\frac{r_1}{a}\right)^m\left[A^i_{mn}F_n(\theta_1)+B^i_{mn}G_n(\theta_1)\right](1-\cos(\theta_2)) \\ &+\sum_{m=0}^{n_r}\sum_{n=0}^{n_\theta}\left(\frac{r_2}{a}\right)^m\left[E^i_{mn}F_n(\theta_2)+D^i_{mn}G_n(\theta_2)\right](1-\cos(\theta_1))] \\ &+\sum_{j=0}^{p}\sum_{k=0}^{p-j}\beta^{u_i}_{jk}x_1^j x_2^k\}, i=1,2 \end{aligned} \quad (1)$$

$$\begin{aligned} \sigma_{ij} &= G_{ij}(x_1,x_2) + H_{ij}(x_1,x_2)Z_{ij}(\theta_1,\theta_2) \\ &\times\{\sqrt{\frac{a^2}{r_1 r_2}}[\sum_{m=0}^{n_r}\sum_{n=0}^{n_\theta}\left(\frac{r_1}{a}\right)^m\left[A^{ij}_{mn}F_n(\theta_1)+B^{ij}_{mn}G_n(\theta_1)\right](1-\cos(\theta_2)) \\ &+\sum_{m=0}^{n_r}\sum_{n=0}^{n_\theta}\left(\frac{r_2}{a}\right)^m\left[E^{ij}_{mn}F_n(\theta_2)+D^{ij}_{mn}G_n(\theta_2)\right](1-\cos(\theta_1))] \\ &+\sum_{k=0}^{p}\sum_{l=0}^{p-k}\beta^{\sigma_{ij}}_{kl}x_1^k x_2^l\}, i=1,2 \end{aligned} \quad (2)$$

$$F_n(\theta)=\sin\frac{n+1}{2}\theta, G_n(\theta)=\cos\frac{n}{2}\theta, n=1,2,... \qquad Z_{ij}(\theta)=1+\cos\frac{\theta_1+\theta_2}{2}, i=1,2, j=2 \quad (3)$$

*Corresponding author E-mail: kamali@hormozgan.ac.ir

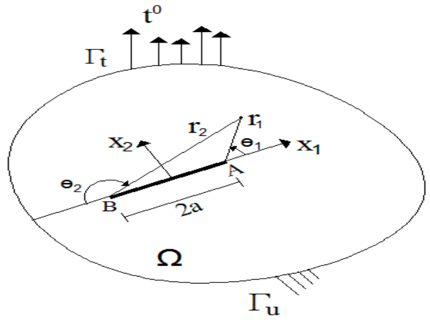

Fig. 1: An anisotropic homogeneous elastic solid with an interior crack

The roles of functions $B_i(\boldsymbol{x})$, $F_i(\boldsymbol{x})$, $H_{ij}(\boldsymbol{x})$, and $G_{ij}(\boldsymbol{x})$ are to satisfy the essential homogeneous, essential inhomogeneous, natural homogeneous, and natural inhomogeneous boundary conditions, respectively. The functions $H_{ij}(\boldsymbol{x})$, account for satisfaction of traction free boundary conditions on the crack faces. Similarly, for the case of an edge crack, the formulation can be obtained.

Using the Reissner's variational principle [4], the governing variational equation is obtained:

$$\delta\iint\left(\frac{1}{2}\varepsilon_G^T\sigma_M+\sigma_M^T(\varepsilon_G-\varepsilon_H)\right)dA-\delta L_e=0 \qquad (4)$$

where subscripts G and H are employed for denoting strains obtained from strain-displacement relation and Hook's law, respectively. Subscript M is used for stresses obtained from Eq. (2), and L_e is the work of external forces. Eq. (4) with the aid of Eqs. (1)-(3) leads to a system of linear equations.

3. Results and Discussion

To demonstrate the robustness of the present methodology, an orthotropic cantilevered rectangular plate with an edge crack as shown in Fig. 2, is considered. The dimensions of the plate are $W=7$ units and $H=16$ units and the crack length is $a=W/2$. In the case where the plate is isotropic, the trends

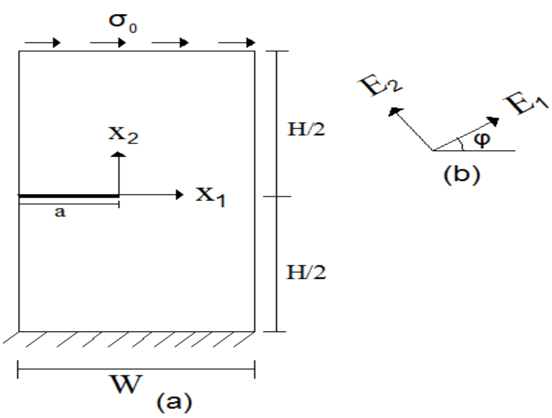

Fig. 2. An orthotropic rectangular plate with an edge crack under uniform shear stress: (a) geometric details, (b) fiber orientation.

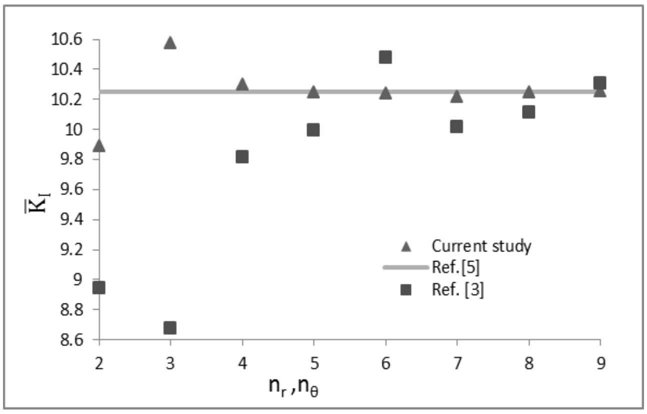

Fig. 3. Trend of convergence of value of $\bar{K}_I$ based on the present method and the work of Ref. [3]

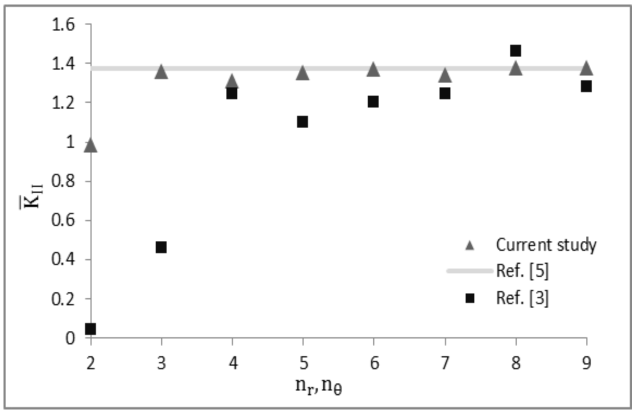

Fig. 4. Trend of convergence of value of $\bar{K}_{II}$ based on the present method and the work of Ref. [3]

Table 1. Value of normalized SIF, $\bar{K}_I$ for different values of fiber orientation, φ

φ	[7]	[8]	Current study	Relative error with Ref. [6] (%)	Relative error with Ref. [7] (%)
0	8.695	8.958	8.7059	0.12	-2.8
10	8.787	9.047	8.8176	0.35	-2.5
20	9.014	9.219	9.0384	0.27	-1.9
30	9.278	9.381	9.2949	0.18	-0.9
40	9.475	9.513	9.4939	0.2	-0.2
50	9.451	9.462	9.4709	0.21	0.09
60	9.305	9.294	9.3277	0.24	0.36
70	9.114	9.124	9.1315	0.19	0.08
80	8.943	9.957	8.9558	0.14	-10.0
90	8.866	8.867	8.8796	0.15	0.14

Table 2. Values of normalized SIF, $\bar{K}_{II}$ for different values of fiber orientation, φ

φ	[7]	[8]	Current study	Relative error with Ref. [6] (%)	Relative error with Ref. [7] (%)
0	1.358	1.356	1.3621	0.3	0.45
10	1.364	1.372	1.3703	0.45	-0.12
20	1.395	1.403	1.3988	0.27	-0.3
30	1.432	1.431	1.4375	0.38	0.45
40	1.438	1.438	1.4459	0.55	0.54
50	1.432	1.417	1.4368	0.32	1.39
60	1.383	1.369	1.3874	0.31	1.34
70	1.260	1.242	1.2643	0.34	1.79
80	1.109	1.093	1.1124	0.3	1.77
90	1.037	1.024	1.0401	0.29	1.57

of convergence of normalized Stress Intensity Factors (SIFs), $\bar{K}_I$ and $\bar{K}_{II}$ based on the current study and the formulation of [3] are shown in Figs. 3 and 4, respectively. These figures show that the SIFs values obtained by the present method have higher convergence rate than those obtained based on the previous study [3].

In the case where the plate is orthotropic with elastic constants, $E_1 = 144.8\text{GPa}$, $E_2 = 11.7\text{GPa}$, $\nu_{12} = 0.21$, and $G_{12} = 9.66\text{GPa}$, values of $\bar{K}_I$ and $\bar{K}_{II}$ for different values of fiber orientation are given in Tables 1 and 2, respectively and are compared with those given in references [6] and [7]. These tables show that the results of the present methodology are closer to those of Ref. [6].

4. Conclusion

In this paper, a semi-analytical method based on the Reisner's mixed formulation is presented for the elastic analysis of anisotropic homogeneous solids with an edge or interior crack. The features of the present methodology are the exact account of the: (1) singularity behavior of the stress field at the crack-tip, (2) discontinuity of the displacement field across the crack faces, (3) satisfaction of the essential homogeneous and inhomogeneous boundary conditions, and (4) satisfaction of the natural homogeneous and inhomogeneous boundary conditions. The SIF values obtained by mixed formulation have a higher convergence rate than those obtained based on the displacement formulation.

References

[1] D. Motamedi, S. Mohammadi, Dynamic crack propagation analysis of orthotropic media by the extended finite element method, International Journal of Fracture, 161(1) (2010) 21-39.

[2] M. Fleming, Y. Chu, B. Moran, T. Belytschko, Enriched element free Galerkin methods for crack tip fields, International Journal for Numerical Methods in Engineering, 40(8) (1997) 1483-1504.

[3] M.T. Kamali, H.M. Shodja, An accurate semi-analytical method for an arbitrarily oriented edge or interior crack in an anisotropic homogeneous elastic solid, European Journal of Mechanics-A/Solids, 45 (2014) 133-142.

[4] E. Carrera, Evaluation of layerwise mixed theories for laminated plates analysis, AIAA journal, 36(5) (1998) 830-839.

[4] Y.T. Gu, W. Wang, L.C. Zhang, X.Q. Feng, An enriched radial point interpolation method (e-RPIM) for analysis of crack tip fields, Engineering Fracture Mechanics, 78(1) (2011) 175-190.

[6] S.J. Chu, C.S. Hong, Application of the Jk integral to mixed mode crack problems for anisotropic composite laminates, Engineering Fracture Mechanics, 35(6) (1990) 1093-1103.

[7] P. Sollero, M.H. Aliabadi, Fracture mechanics analysis of anisotropic plates by the boundary element method, International Journal of Fracture, 64(4) (1993) 269-284.