Development of Parametric and Time Dependent Reduced Order Model for Diffusion and Convection-Diffusion Problems Based on Proper Orthogonal Decomposition Method

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Abstract

Simulation and numerical analysis of physical phenomena, especially for unstable problems, due to dependency of the numerical algorithms on the computer hardware to the increasing of the number of computational nodes, is the most important features of their solutions. For this reason, increases the number of computations then increased computational costs. The order reduction method is one that has been widely used in recent years to reduce computational time. In this way, by reducing the constraints of the system, without changing the inherent features of the problem, the computational efficiency will dramatically increase. In this study, using the basic concepts of dynamical systems, two problems of thermal diffusion and convection-diffusion are investigated independently and by using the proper orthogonal analysis method, a reduced order model is established for the equations governing these phenomena created. Accordingly, for each of the problems, based on the projection of the governing equation in the vector space of modes, by using more energetic modes, a reduced order model is obtained with respect to the orthogonal basis properties. The model obtained in order to simulate the process time variations can properly replace the original equation and predict the behavior of the system with very good accuracy.

Key words: Proper Orthogonal Decomposition, Diffusion Equation, Convection-Diffusion Equation, Reduced Order Model, Surrogate Model

1 - Introduction

proper orthogonal decomposition (POD) is one of the most common ways to reduce the order of the problem [1]. The POD method was first studied by Karhunen-Loeve in 1946 [2]. For the first time in 1967, Lamley suggested that POD could be used to extract large structures appearing in turbulent flows and emitting radio waves [3]. Subsequently, due to the limitations of computer hardwares and numerical models, this method remained useless for a long time. In the late 1980s, with the advent of snapshots method by Sirovich, POD was introduced as an efficient tool for developing reduced order models for dynamical systems and fluid dynamics problems [4].

2 - POD– snapshots Method

For using the POD-snapshots method, a sequence of fluctuations data will be arranged as a snapshots ensemble. Then by solving the eigenvalue problem for the snapshots matrix, proper orthogonal bases will be computed.

3 - Galerkin Projection and Dynamical System Equation

in order to develop the reduced order model, by using Galerkin projection of the governing equation, the dynamical system For each of the problems as thermal diffusion and convection-diffusion problems, these equations are as follows:

$$\frac{da^k(\mathrm{t})}{dt} = B_i^k \times \mathrm{a}^i(\mathrm{t}) + \mathrm{C}^k \tag{1}$$

$$\frac{da^k(t)}{dt} = A_{ij}^k \times a^i(t) \times a^j(t) + B_i^k \times a^i(t) + C^k \tag{2}$$

4 – Selection of the number of modes to reconstruct the field

The number of modes which are captured a high level of kinetic energy of flow field , is calculated by the following equation:

$$k = \frac{\sum_{i=1}^{N_r} S_i^2}{\sum_{i=1}^{N_{total}} S_i^2} \tag{3}$$

where, S_i are the singular values of the snapshot matrix and N_r is the required number of modes for reconstruction of reduced order model.

5 - Results and discussion

In the first problem, by solving the 2D transient diffusion equation for the diffusion coefficient of 0.0044 over a time interval of 0.75, a snapshots ensemble with 75 members is provided. Then, by solving the singular value problem for snapshot matrix, the vector space containing the modes is obtained. In this investigation, the major portion of the kinetic energy in modes (99.7% of total energy) has been extracted by 4 modes. Then, by using Runge-Kutta, equation **1** over time interval of 0.75 with the time step of 0.001 is solved to calculate the modal coefficients variations. In order to validate the results of the reduced order model, in Fig. **1**, the outcomes will be compared with the results of the direct numerical simulation. Based on the high accuracy of prediction of modal coefficients in the short time interval of 0.75 for diffusion coefficient of 0.0044, a surrogate model to predict the field dynamics has been achieved. Using this parametric model, it is possible to obtain the field variation in the short time interval of 0.75 for different values of the thermal diffusion coefficients. In Fig. **2**, the temperature distribution over the vertical center line in the last time step and for the diffusion coefficients of 0.007 are shown.

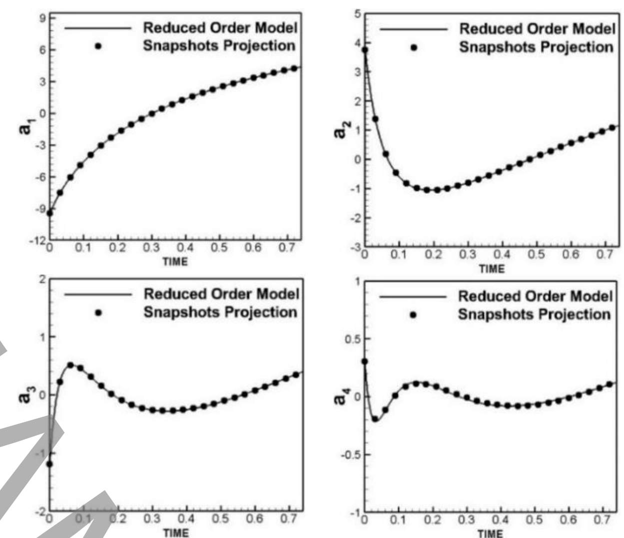

Fig. 1. Comparison Between Time history of Modal coefficients obtained from Reduced Order Dynamical System and Results of Snapshots Projection

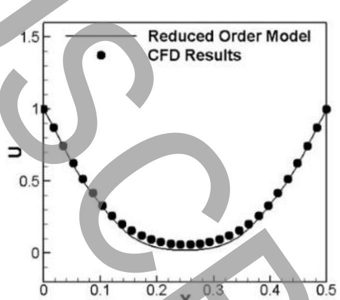

Fig. 2. Comparison between Temperature Distribution for the Last Time Step in Vertical Center line for Diffusion Coefficients of 0.007

In the second problem, by solving the burgers equation at time interval of 4 at Re=100, a snapshots ensemble with 80 members is considered. Then, by solving the singular value problem, the vector space containing the modes is obtained. In this study, the major portion of the kinetic energy in modes (99.3% of total energy) has been extracted by 4 modes. Then, equation **2** is solved with time step of 0.001 and for Re=100 to calculate the modal coefficient variations. In Fig. **3**, outcomes compared with DNS.

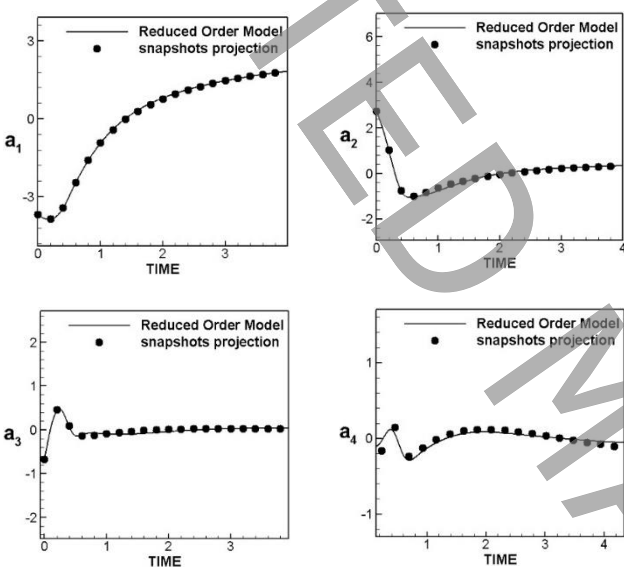

Fig. 3. Comparison Between Time history of Modal coefficients obtained from Reduced Order Dynamical System and Results of Snapshots Projection

Due to the high accuracy of predicting the time variation of the Burgers equation (modal coefficients) in the short time interval of 4 units and for the Re=100, a surrogate model was obtained. Using this time-dependent model, the variations of the field at different time interval and Re=100 can be calculated. In Fig. **4**, the distribution of the function of the Berger equation is shown in the last time step at 10,000 time steps.

6 - Conclusion

POD is a powerful tool for reducing the cost of computations. By using of POD and transformation of the governing equations to the vector space consisting of basic vectors, a new form the governing equation is created. Next, more energetic modes were obtained from the initial snapshots ensemble and thus by developing a reduced order model, the field dynamics with fewer dimensions has been carefully reconstruction. The results of the reduced order model are compared with the relative direct numerical simulation and show high accuracy and appropriate capabilities of this method. As a result, it is possible to develop accurate and fast models based on the basic concepts of machine learning methods.

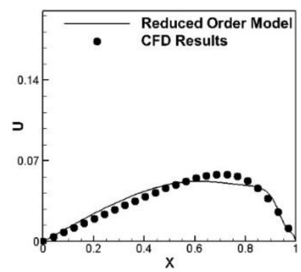

Fig. 4. Comparison Between Response of Burgers Equation in X-Direction for the Last Time Step for 10,000 Time Step

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