Analytical Modeling of Elastic Limit Angular Velocity in A rotating Disk of A functionally Graded Material under Mechanical-Thermal Loading Conditions

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ABSTRACT

Thermal stresses caused by temperature changes, along with high angular velocities in industrial rotating disks will reduced the strength of the disk material. Therefore, analysis of rotating disks under thermal-mechanical loads and estimation of the elastic limit angular velocity have particular importance as a criterion of the initiation of plastic deformation. In this paper, analytical modeling for thermoelastic analysis of functionally graded rotating disks is performed by considering the variations of all the geometric and mechanical properties of the rotating disk in a radial direction. The homotopy perturbation method is used as an analytical method to solve equations. The results are verified by the finite difference method and the data in the references. Numerical analysis is performed to investigate the influence of thickness parameter, thermal loading type and boundary conditions on the limit angular velocity and the radius of initiation of the plastic deformation. The Tamura-Tomota-Ozawa model is used to calculate yield stress at different radius of functionally graded disk. Finally, it is shown that by defining the appropriate temperature gradient on the outer surface as a boundary condition, the level of thermal stresses can be controlled and reduced up to 20% compared to the constant thermal boundary condition.

KEYWORDS

Rotating disk, functionally graded material, Homotopy perturbation method, Yield stress, limit angular velocity

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1. Introduction

Rotating disks are mechanical components that are used in a wide range of mechanical equipment. It is recommended to use disks with non-uniform thickness profiles, made of functionally grade materials.

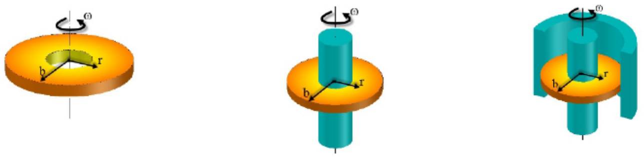
One of the first researches on rotating disks was the analytical solution of homogeneous elastic-plastic disks by Gamer using tresca yielding theory [1]. The elastic-plastic deformation of a homogeneous rotating disk of variable thickness with three types of boundary conditions as free, under pressure and under constraint in the radial direction is done by Araslan [2].The effect of ductile material failure models on the behavior of plastic deformations of rotating disks with variable thickness was investigated for the first time by Akbari et al. [3]. Araslan was one of the first researchers that solve the non-isothermal governing equations of rotating disks in both elastic and plastic states [4]. Zharfi studied the creep behavior of functional graded rotating disks by considering nonlinear and non-uniform distributions for mechanical properties and concluded that external loads affect the creep rate in the rotating disk [5].

To achieve a reliable solution, an analytical model for studying the thermoelastic behavior of functionally graded rotating disks with variable thickness in general and for different types of thermal loading and boundary conditions is presented. The governing equations will be solved using the analytical homotopy perturbation method. The von Mises criterion is used to calculate the equivalent stresses in the disk. The angular velocity of the disk will be used as a criterion to limit the maximum stress. The model proposed by Tamura-Tomota-Ozawa is used to calculate the yield stress at different radius of the rotating disk.

2. Mechanical-thermal modeling of a rotating disk

2.1. Rotating disk boundary conditions

In accordance with the real working conditions of rotating disks, analytical modeling and presentation of results in this paper for three different boundary conditions in the inner and outer surfaces of rotating disks are performed according to Figure 1.



a: Free-Free | b: Radially constrained-Free | c: Radially constrained Radially constrained

Figure 1. Rotating disk with different boundary conditions on inner and outer surfaces

2.2. Mechanical properties

In functionally graded rotating disks, by considering a proper mixture law, the mechanical properties can be changed as a function of the volume fraction, between the properties of the ceramic and the metal from the inner surface to the outer surface. The mechanical properties of these materials are given in Table 1.

Table 1. Mechanical properties of metal and ceramics in a functionally graded rotating disk

Material properties	Aluminum	Zirconia
Young modulus (GPa)	70	151
Poisson ratio	0.3	0.3
Density	2700	5700
Thermal expansion coefficient	23	10
Displacement conductivity	209	2
Yeild stress (MPa)	300	---
Tangent modolue (GPa)	35	---
Constant *q* (GPa)	7.5	

According to Table 2, the parameters m_1, m_2, m_3, m_4 are known as the indexing parameters of these mechanical properties in the structure of functional graded materials and m_5 as the parameter of disk thickness change.

Table 2. Mechanical properties of functionally graded materials by defining its parameters

Displacement conductivity	Thermal expansion coefficient	Density	Young modulus
$k(r)=k_e\left(r/r_o\right)^{m_4}$	$\alpha(r)=\alpha_e\left(r/r_o\right)^{m_3}$	$\rho(r)=\rho_e\left(r/r_o\right)^{m_2}$	$E(r)=E_e\left(r/r_o\right)^{m_1}$
$k_e=k_o$	$\alpha_e=\alpha_o$	$\rho_e=\rho_o$	$E_e=E_o$
$m_4=\frac{Ln\,(k/k_o)}{Ln\,(r_i/r_o)}$	$m_3=\frac{Ln\,(\alpha_i/\alpha_o)}{Ln\,(r_i/r_o)}$	$m_2=\frac{Ln\,(\rho_i/\rho_o)}{Ln\,(r_i/r_o)}$	$m_1=\frac{Ln\,(E_i/E_o)}{Ln\,(r_i/r_o)}$

2.3. Equivalent stress

The yield stress analysis in graded materials is a different from that of homogeneous materials, and like other properties in these materials, the yield stress is different in different parts. In this case, the yield stress in the rotating disk can be calculated using an experimental parameter called q as the following relation:

$$q=\frac{\sigma_c-\sigma_m}{\varepsilon_c-\varepsilon_m} \tag{1}$$

Based on this parameter, the yield stress based on the Tamura-Tomota-Ozawa model can be calculated from the following equation:

$$\sigma_y(r)=\sigma_{ym}\left(\left(1-V_c(r)\right)\frac{q+E_m}{q+E_c}\frac{E_c}{E_m}V_c(r)\right) \tag{2}$$

2.4. Thermal modeling

The governing equation of heat transfer in the rotating disk, regardless of the heat source, and having the displacement coefficient ($k(r)$) and the thickness profile ($h(r)$) is expressed as follows:

$$\frac{d}{dr}\left(rh(r)k(r)\frac{dT(r)}{dr}\right)=0 \quad (3)$$

Thermal boundary conditions on the inner and outer surfaces of the rotating disk can be defined in general terms [13]:

$$C_{11}T(r_i)+C_{12}\left.\frac{dT(r)}{dr}\right|_{r=r_i}=\xi_i \quad (4)$$

$$C_{21}T(r_o)+C_{22}\left.\frac{dT(r)}{dr}\right|_{r=r_o}=\xi_o \quad (5)$$

- Thermal condition **A**: The temperature on the inner surface T_i and on the outer surface T_o is constant.
- Thermal condition **B**: The temperature of the inner surface T_i is constant and the temperature gradient of the outer surface is equal to T_o.

2.5. Mechanical modeling

The Navier equation governing rotating disks based on the displacement in the radial direction are:

$$\frac{d^2u_r(r)}{dr^2}+\left(\frac{1}{r}+\frac{1}{h(r)}\frac{dh(r)}{dr}+\frac{1}{E(r)}\frac{dE(r)}{dr}\right)\frac{du_r(r)}{dr}+\left(\frac{v}{rh(r)}\frac{dh(r)}{dr}+\frac{v}{rE(r)}\frac{dE(r)}{dr}-\frac{1}{r^2}\right)u_r(r)+\left(-\frac{1}{h(r)}\frac{dh(r)}{dr}-\frac{1}{E(r)}\frac{dE(r)}{dr}-\frac{1}{\alpha(r)}\frac{d\alpha(r)}{dr}\right)\alpha(r)(1+v)T(r)=-\frac{(1-v^2)\rho(r)\omega^2 r}{E(r)} \quad (6)$$

3. Homotopy perturbation method [6]

Based on the homotopy technique, the homotopy function can be formulated to satisfy the following equations:

$$H(v,p)=L(v)-L(u_o)+pL(u_o)+p\left[N(v)-f(r)\right]=0 \quad (7)$$

In this relation, N is the linear part and L is the nonlinear part of the differential equation. The answer to Eq. (7) can be written as a series of powers:

$$v=v_0+pv_1+p^2v_2+p^3v_3+\ldots \quad (8)$$

With replace Eq. (8) in Eq. (7) and arrange the obtained relation according to different powers (p). Each of the power coefficients of a differential equation will be solved and finally the answer of equation is:

$$u=\lim_{p\to 1} v=v_0+v_1+v_2+v_3+\ldots \quad (9)$$

4. Discussion and Results

Table 3 shows the results obtained from homotopy perturbation and finite difference method to solve the Navier equation at different radius of the disk.

Table 3. Comparison of finite difference method and homotopy perturbation method for the thickness parameter ($m_5=-0.5$)

Disk radius (m)	percentage error	percentage error
	Thermal condition ***A***	Thermal condition ***B***
0.2	0.74%	0.68%
0.3	0.52%	0.45%
0.4	0.37%	0.25%
0.5	0.24%	0.19%
0.6	0.17%	0.17%
0.7	0.083%	0.11%
0.8	0.077%	0.102%
0.9	0.073%	0.1007%
1	0.035%	0.049%

The exact values of the angular velocity of the elastic are observed for different values of the thickness parameter, the disk boundary condition and the thermal boundary condition, in Table 4.

Table 4. The results for different values of thickness parameter (m_5)

Disk radius (m)		Free-Free		Free-Radially constrained	
		condition *A*	condition *B*	condition *A*	condition *B*
Angular velocity	$m_5=0$	78	259	121	274
	$m_5=-0.5$	205	323	312	404
	$m_5=-1$	315	402	405	493

5. Conclusions

According to the results, it can be concluded that the results presented by two methods have a very good accuracy of both heat and stress analysis. Numerical analysis showed that the disk with B thermal condition experiences less stress compared to the A thermal boundary condition for different types of disk boundary conditions and different values of the thickness parameter m_5. Therefore, by defining the appropriate temperature gradient on the outer surface of the disk as the B thermal condition, the level of thermal stresses in it can be controlled and optimized.

6. References

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