# Free vibration and flexural-torsional stability analyses of axially functionally graded tapered thin-walled beam resting on elastic foundation

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**ABSTRACT**

**The thin-walled beams are widely adopted in different structural components ranging from civil engineering to aeronautical applications due to their conspicuous characteristics. A slender thin-walled beam loaded initially in compression may buckle suddenly in flexural–torsional mode since its torsional strength is much smaller than bending resistance. In this paper, flexural-torsional stability and free vibration analyses of axially functionally graded tapered I-beam resting on Winkler elastic foundation are assessed. Considering the coupling between the flexural displacements and the twist angle, the motion equations are derived via Hamilton's principle in association with Vlasov's thin-walled beam theory. The differential quadrature method is applied to solve the system of differential equations and to acquire the critical buckling loads and natural frequencies. To validate the obtained results, at first, homogeneous tapered I-beam in the absence of elastic foundation was analyzed and compared with finite element solution using ANSYS and other available benchmarks. Afterward, the numerical outcomes for axially graded non-prismatic I-beam resting on elastic foundation are reported in graphical form to find out the impacts of axial load position, beam's length, end conditions, web and flanges tapering ratio, material gradient index, Winkler parameter and spring position on the non-dimensional buckling loads and vibration frequencies.**

**KEYWORDS**

**Flexural-torsional buckling, Vibration frequency, Functionally graded materials, Winkler foundation, Differential quadrature method**

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## 1. Introduction

In the recent years, by improvements in fabrication process, structural members have been possible with mixed materials such as wood, steel and composite. Functionally Graded Materials (FGMs) are a new class of advanced materials made up by gradually and smoothly changing the composition of two or more different materials in any desired direction. The use of non-prismatic elements made up of FGMs during the past twenty years has been increasing in complicated mechanical components such as turbine blades, rockets, aircraft wings and space structures due to their conspicuous characteristics such as high strength, thermal resistance and optimal weight distribution. Regarding this, in this paper, the flexural-torsional free vibration and stability analyses of axially functionally graded web and/or flanges tapered I-beam resting on Winkler foundation are investigated using the differential quadrature method (DQM).

## 2. Governing Equations

As shown in Fig.1, a straight tapered thin-walled beam resting on Winkler foundation is taken into account. The right hand Cartesian co-ordinate system, with *x* the initial longitudinal axis measured from the left end of the beam, the *y*-axis in the lateral direction and the z-axis along the thickness of the beam is considered. The origin of these axes (*O*) is located at the centroid of doubly-symmetric I-section.

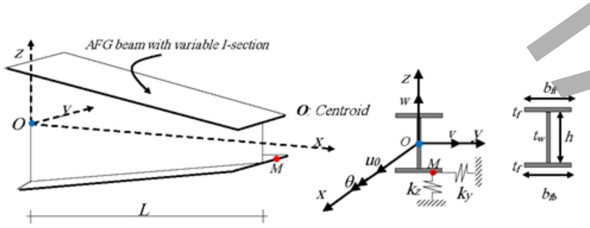


**Fig. 1: (a) Thin-walled beam with doubly-symmetric cross-section, (b) Coordinate system and notation of displacement parameters**

Based on the Vlasov model [1] for a non-uniform torsional loading condition, in the context of small displacements, the displacement field for an arbitrary point of the beam can be expressed as follows

$$U(x,y,z)=u(x)-y\,\frac{dv(x)}{dx}-z\,\frac{dw(x)}{dx} -\omega(y,z)\frac{d\theta(x)}{dx}$$
$$V(x,y,z)=v(x)-z\,\theta(x) \qquad (1)$$
$$W(x,y,z)=w(x)+y\,\theta(x)$$

where $U,V,W$ stand for to the axial, lateral and vertical displacement components along the $x,y,z$ direction, respectively, whereas $u,v,w$ are the kinematic quantities defined at the reference surface, the term $\omega(y,z)$ refers to the warping function for the variable cross-section, defined by means of Vlasov torsion theory [1], and $\theta$ is the twisting angle. The equations of motion and end conditions for the free vibration of AFG tapered I-beam resting on uniform Winkler foundation are derived using Hamilton's principle []. In the absence of external forces, the principle can be written as:

$$\int_{t1}^{t2}\delta\Pi dt=\int_{t1}^{t2}\delta\left(U_l+U_0+U_f-U_M\right)dt=0$$
$$\delta\Pi=\delta U_l+\delta U_0+\delta U_f-\delta U_M=0 \qquad (2)$$

where $t_1$ and $t_2$ are two arbitrary time variables, and $\Pi$ is the total potential energy. $\delta$ denotes a virtual variation. $U_l$ and $U_0$ are the elastic strain energy and the strain energy due to effects of the initial stresses, $U_M$ the kinetic energy under harmonic forces, and $U_f$ is the energy corresponding to a uniform elastic foundation. Their relationships for each term of the total potential energy are developed separately in the following:

$$\delta U_l=\int_0^L\int_A\left(\sigma_{xx}\,\delta\varepsilon_{xx}^l+\tau_{xy}\,\delta\gamma_{xy}^l+\tau_{xz}\,\delta\gamma_{xz}^l\right)dA\,dx$$
$$\delta U_0=\int_0^L\int_A\left(\sigma_{xx}^0\,\delta\varepsilon_{xx}^*+\tau_{xy}^0\,\delta\gamma_{xy}^*+\tau_{xz}^0\,\delta\gamma_{xz}^*\right)dA\,dx$$
$$\delta U_M=\omega^2\int_L\int_A\rho\left(U\,\delta U+W\,\delta W+V\,\delta V\right)dA\,dx \qquad (3)$$
$$\delta U_f=\int_L\left(k_y v_M\,\delta v_M+k_z w_M\,\delta w_M\right)dx$$

where $L$ and $A$ stand for the beam length and the cross-sectional area, respectively, $\rho$ and $\omega$ indicate the material density and the natural frequency (circular). Moreover, ($\delta\varepsilon_{xx}^i,\delta\gamma_{xz}^i,\delta\gamma_{xy}^i$) and ($\delta\varepsilon_{xx}^*,\delta\gamma_{xz}^*,\delta\gamma_{xy}^*$) refer to the variation of the linear and non-linear part of the strain tensor, respectively; whereas $\sigma_{xx},\tau_{xz},\tau_{xy}$ denote the Piola–Kirchhoff stress tensor components, and $\sigma_{xx}^0,\tau_{xz}^0,\tau_{xy}^0$ are the initial axial and shear stress conditions. In (3), $k_y$ and $k_z$ denote Winkler foundation modulus for the lateral and transverse translations at the point **M**. Regarding Eq. (1), the two components of vertical and lateral displacements at point **M** can be found as follows:

$$v_M=v-h_z\theta \qquad w_M=w+h_y\theta \qquad (4)$$

In this study, it is contemplated that the concentrated compressive axial load (*P*) is applied at end beam without an eccentricity from centroid along z-axis. Therefore, an external bending moment occurs about the major principal axis ($M_y^*$) and the magnitude of bending moment with respect to z-axis ($M_z^*$) is equal to zero. Regarding this, the most general case of normal and shear stresses associated with the external bending moment $M_y^*$ and shear force $V_z$ are considered as

$$\sigma_{xx}^0 = \frac{P}{A} - \frac{M_y^*}{I_y} z \;,\quad \tau_{xz}^0 = \frac{V_z}{A} = -\frac{M_y^{*\prime}}{A},\quad \tau_{xy}^0 = 0 \qquad (5)$$

Based on the assumption of a Green's strain-tensor, the linear and non-linear parts of the kinematic relations are as follows [2, 3]

$$\begin{aligned}
\varepsilon_{xx}^l &= u_0' - yv'' - zw'' - \omega\theta'' \\
\gamma_{xz}^l &= 2\varepsilon_{xz}^l = \left(y - \frac{\partial \omega}{\partial z}\right)\theta' \\
\gamma_{xy}^l &= 2\varepsilon_{xy}^l = -\left(z + \frac{\partial \omega}{\partial y}\right)\theta' \\
\varepsilon_{xx}^* &= \frac{1}{2}\left[v'^2 + w'^2 + r^2\theta'^2\right] + yw'\theta' - zv'\theta' \\
\gamma_{xz}^* &= -(v' + \theta' z)\theta \\
\gamma_{xy}^* &= (w' + \theta' y)\theta
\end{aligned} \qquad (6)$$

Based on the equation presented above, the first variation of the total potential energy contains the virtual displacements ($\delta u$, $\delta v, \delta w, \delta\theta$) and their derivatives. After appropriate integrations by parts, and mathematical simplifications, we get the following governing equations of motion:

$$\begin{aligned}
&\left(EAu_0'\right)' + \rho\omega^2 A u_0 = 0 \\
&\left(EI_z v''\right)'' - Pv'' - (M_y v)'' - \rho\omega^2 A v \\
&+\omega^2\left(\rho I_z v'\right)' + k_y v - k_y h_z \theta = 0 \\
&\left(EI_y w''\right)'' - Pw'' + (M_z w)'' - \rho\omega^2 A w \\
&+\omega^2\left(\rho I_y w'\right)' + k_z w + k_z h_y \theta = 0 \\
&\left(EI_\phi \theta''\right)'' - \left(GJ\theta'\right)' - P(r_c^2\theta')' + M_z w'' \\
&-M_y v'' - \rho\omega^2 I_c \theta(x) + \omega^2\left(\rho I_\phi \theta'\right)' \\
&-k_y h_z v + k_y h_z^2\theta + k_z h_y w + k_z h_y^2\theta = 0
\end{aligned} \qquad (7)$$

## 3. Results and Discussion

To solve the coupled equations of motion and, the DQM is employed to calculate the natural frequency and critical buckling load of AFG web and flanges tapered I-beam resting on uniform Winkler foundation subjected to different end conditions. In order to validate the acquired outcomes of methodology presented herein, comparisons have been carried out with those estimated via a finite element formulation by Soltani et al. [3], and ones obtained by ANSYS software [4]. In the case of simply supported member, the non-dimensional flexural-torsional buckling load parameter variation versus the power-law exponent for different values of elastic foundation constants where in numerical computations $\alpha=\alpha_1=0.5$ is selected and load is applied on the top flange of right end section, is presented in Fig. 2.

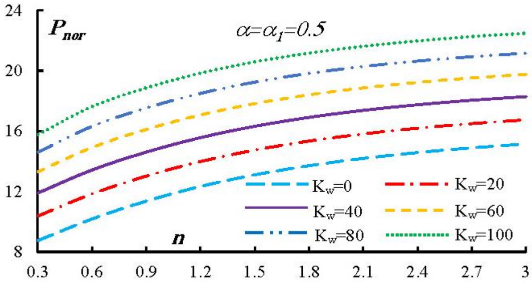


**Fig. 2: The effect of Winkler foundation modulus and power-law index on critical buckling load**

Next, the influence of Winkler parameters (ranging from 0 to 90) on the variations of the non-dimensional natural frequency ($\omega_{nor}$) of simply supported thin-walled beam with varying I-section made up of homogeneous material and axially functionally one (n=1.5) with respect to tapering ratios (varying from 0 to 1.0) is plotted in Figs. 3. In this stage, the non-uniform beam having equal web height and flanges width tapering ratios ($\alpha=\alpha_1$) is perused.

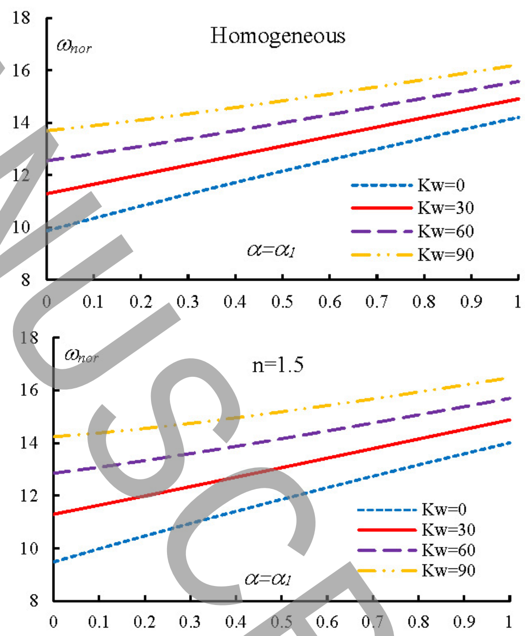


**Fig. 3: Effect of tapering ratios on non-dimensional natural frequency of simply supported beam for different values of Winkler foundation modulus and gradient indexes**

## 4. Conclusions

The present research deals with the flexural-torsional buckling and free vibrational analyses of AFG tapered doubly-symmetric I-beam resting on elastic foundation. Based on Vlasov's theory for thin-walled cross-section, the governing equations of motion are derived via the Hamilton principle. The effect of axial load eccentricity is also considered in the formulation. The differential quadrature method is then used to estimate the buckling load and natural frequency for web and flanges tapered beam. According to the obtained numerical outcomes, it is concluded that that for both uniform and non-uniform I-beams and all values of Winkler foundation constants, as non-homogeneity parameter increases the stability strength and enhances. It can also be stated that for $0.5 \le n \le 1.5$, the non-dimensional critical load and vibrational frequency parameters increase obviously whereas, for $n$>1.5, these parameters increase slightly and approaches maximum magnitude. It is also found out that for any value of power-law exponent and Winkler parameter, the buckling load and natural frequency of prismatic beam and double tapered one with is least and most, respectively. The numerical outcomes show that the elastic foundation increases the stability and vibrational characteristics of axially non-homogeneous and homogeneous I-beams with constant or variable cross-section.

**5. References**

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