Approximate torsional analysis of arbitrary trapezoidal bars by Kantorovich method

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ABSTRACT

A variety of structural members are expected to safely tolerate torsional moments. These include irregularly-shaped cross sections (e.g., trapezoidal or triangular sections) in some industries which deserve special considerations for the analysis and design under torsional loading. Therefore, the development of novel methods as alternative approaches seems very necessary, partially because of the deficiency of analytical solution in treating asymmetric solution domains. Semi-analytical and numerical methods appear as desirable alternatives in most cases. One of the proper tools for dealing with the boundary value problems encountered in torsional analysis is the vibrational method. The Kantorovich semi-analytical method, known as an extension of the Rayleigh-Ritz method, has been proven advantageous among the others, mainly because of relaxing the conventional limitations in selecting the primary function for satisfying the boundary conditions. Therefore, the purpose of the present study is to extend the applicability of Kantorovich method to estimate the warping and stress field of arbitrary trapezoidal sections directly. Finally, the efficiency and accuracy of the present solution is verified against a number of existing analytical and numerical methods. The results indicate high precision and rapid convergence of this semi-analytical method.

KEYWORDS

Torsion problem, Kantorovich method, trapezoidal sections, warping function, Prandtl's stress distribution.

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1. Introduction

Many engineering members, such as beams, shafts, and wings, are subjected to torsional moments, so understanding the torsional behavior of these members is critical to analyzing and designing them. The solution to the first torsion problem for circular sections was proposed by Coulomb in 1784. Accordingly, the circular cross section remains planar after torsional moment is applied. However, describing the torsional behavior was far more difficult for non-circular sections and remained unsolved for a long time. This difficulty is due to the fact that non-circular sections will no longer remain planar in response to a torque and a complex phenomenon called warping of the cross section occurs in such cases. Saint-Venant was the first to propose the exact formulation for the torsion of prismatic beams of arbitrary cross section on the basis of a semi-inverse approach. Later on, Prandtl introduced the concept of stress function which is governed by the Poisson equation.

In addition to the analytical and numerical methods adopted for the torsional analysis, there are other methods called semi-analytical methods. These enables solving boundary value problems defined in arbitrarily-shaped domains. In the context of variational approach, some approximate or semi-analytical methods have been proposed for solving various types of differential equations involving those related to elasticity. Historically, the credit goes back to the work of Ritz followed by Galerkin and Kantorovich [1]. The method of Kantorovich [1-3] is adopted here, simply because the original partial differential equation is decomposed into one or more ordinary differential equations, resulting in an improved prediction of the solution.

2. Methodology

The following boundary value problem governs the spatial distribution of stress function in the Cartesian coordinates:

$$\nabla^2 \chi = -2 \tag{1}$$

subjected to $\chi = 0$ along the boundary of cross section. The trapezoidal geometry of cross-section (Ω) is defined by $\{a \le x_1 \le b, -m_1 x_1 \le x_2 \le m_2 x_1\}$. Such geometrical entities render conventional solution techniques powerless in dealing with intended boundary-value problem. To overcome this difficulty, the variational approach becomes more effective in dealing with domains confined by irregular boundaries. First, the associated penalty function is formed as:

$$I(\phi) = \iint [(\frac{\partial \phi}{\partial x_1})^2 + (\frac{\partial \phi}{\partial x_2})^2 - 4\chi]\, dx_1 dx_2 \tag{2}$$

It can be shown that minimizing Eq. (2) is mathematically equivalent to obtaining the solution of Eq. (1). For the present applications, an admissible two-term solution (Eq. (4)) is adopted to set up Euler-Lagrange differential equations in terms of unknown functions f_i, $i = 1,2$ [3]

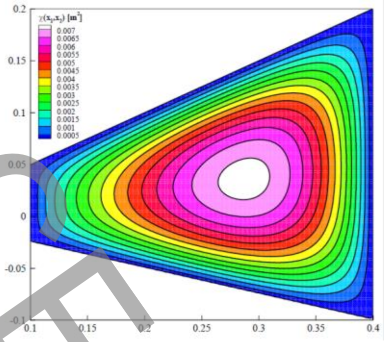
$$\begin{aligned}\bar{\chi} &= \sin\left(\pi \frac{(x_2 + m_1 x_1)}{(m_1 + m_2) x_1}\right) f_1(x_1) \\ &+ \sin\left(3\pi \frac{(x_2 + m_1 x_1)}{(m_1 + m_2) x_1}\right) f_2(x_1)\end{aligned} \tag{3}$$

It is clear that the trigonometric functions in (3) meet the requirement of $\chi = 0$ along the inclined boundaries of the cross section. The remaining boundary conditions are expected to be satisfied by the functions f_1 and f_2. This is regarded as a major advantage of the Kantorovich method in achieving an improved solution to the problem.

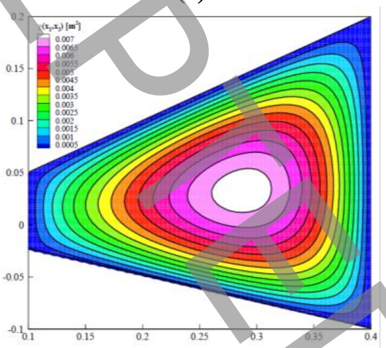
3. Results and Discussion

First, the stress function $\chi(x_1, x_2)$ is depicted in Fig. 1 and the results show a very good agreement between the finite element method and the Kantorovich method. It should be noted that the solution of finite elements has been achieved by calling the pdetool tool in the Matlab environment, whose efficiency in solving similar problems has been proven in previous studies [4]. It is also observed that the maximum values occur in the central areas of the cross section while their lowest values are concentrated near the trapezoidal sides (boundaries of the solution domain). Fig. 2 shows the shear components of the stress tensor normalized with respect to the applied torque T .

The stress field σ_{13}/T reveals odd-symmetry, except that the axis of symmetry is no longer the principal axis x_1 and the contour lines are symmetric with respect to the midline of the vertical sides of the trapezoid. The maximum values of σ_{13} with positive and negative signs also occur in the middle of the lower and upper sides of the trapezoid, respectively (Fig. 2a). Also, the contour line $\sigma_{13} = 0$ coincides with the midline of the vertical sides of the trapezoid. However, the stress field σ_{23}/T shows even-symmetry so that σ_{23} attains its positive maximum value midway on the right face of the cross section (Fig. 2b). The negative maximum value occurs on the opposite side.

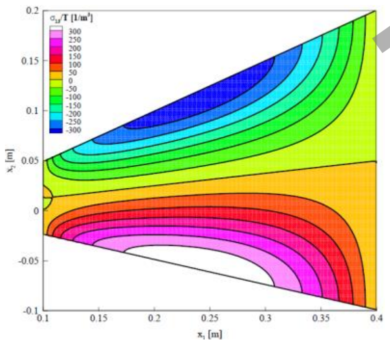


(a)



(b)

Figure 1. Cross-sectional distribution of stress function as computed by (a) present study and (b) finite element method



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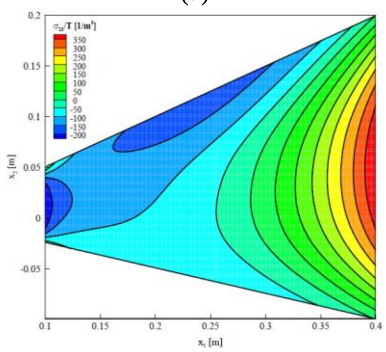

(b)

Figure 2. Cross-sectional distribution of shear stress (a) σ_{13}/T and (b) σ_{23}/T

4. Conclusion

In the present study, a semi analytical method based on variational calculus was proposed to fully investigate the complex behavior of arbitrary trapezoidal-shaped bars under torsion. The solution was approximated as a linear combination of two harmonic terms, satisfying the boundary conditions on inclined faces of the cross section. The stress function and the warping field were obtained in the trapezoidal section. Further, the present solution encompasses the case of triangular bars as a subset, demonstrating well agreement with available analytical and numerical solutions in the cases studied. Extension to the case orthotropic materials is currently underway.

5. References

[1] L.V.e. Kantorovich, Approximate methods of higher analysis, Interscience, (1958).

[2] S.P. Timoshenko, J. Goodier, Theory of elasticity, (2011).

[3] A. Mahdavi, H. Seyyedian, Steady-state groundwater recharge in trapezoidal-shaped aquifers: A semi-analytical approach based on variational calculus, Journal of Hydrology, 512 (2014) 457-462.

[4] A. Mahdavi, M. Yazdani, A novel analytical solution for warping analysis of arbitrary annular wedge-shaped bars, Archive of Applied Mechanics, (2021).