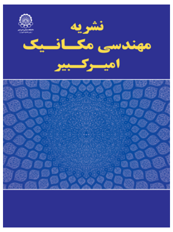

Amirkabir Journal of Mechanical Engineering

Amirkabir J. Mech. Eng., 53(2) (2021) 167-170
DOI: 10.22060/mej.2019.15028.6003

Application of Nonconforming Quadtree Grids in the Finite Element Method

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ABSTRACT: In the standard finite element method, the edges of the adjacent elements are aligned to each other, and the corner of an element does not locate on the edges of another one. If this constraint violates, the mesh is called non-conforming and the use of such meshes in the finite element method requires specific techniques. In the present paper, a new method is suggested for treating non-conforming meshes. Non-conforming meshes appear generally in adaptive mesh refinement processes especially in the quadtree mesh refinement algorithm. The quadtree is a data structure with an extremely fast recursive algorithm and is used to divide a two-dimensional domain into sub-regions or elements. In the present paper, a new approach is proposed to construct the shape functions of such elements. In this method, the shape functions are considered harmonic functions and the Laplace boundary value problem is defined and its solution is used as the shape functions of the non-conforming elements. To evaluate the applicability and accuracy of the proposed method, two numerical examples are solved and the results are presented. The results show that the proposed method can be used to effectively apply the non-conforming meshes in the finite element method.

Review History:

Received: Sep. 24, 2018
Revised: Feb. 06, 2019
Accepted: Mar. 11, 2019
Available Online: Nov. 02, 2019

Keywords:

Quadtree meshes
non-conforming meshes
Finite element method
Shape function

1- INTRODUCTION

Harmonic coordinates are one of the most appealing ideas to drive the approximation functions. The first use of harmonic functions for Two-Dimensional (2D) interpolation was published in Ref. [1]. The first use of harmonic approximation functions for the solution of boundary value problems was appeared in Ref. [2] in which it proofed that such approximation functions form a partition of unity and exactly reproduce a linear field. These functions also possess the Kronecker delta property. Harmonic functions satisfy the elliptic Laplace equation with positive boundary conditions. Therefore, the approximation functions are non-negative in the element interior. To the best knowledge of the authors, a few papers have been published based on the application of harmonic shape functions in the Finite Element Method (FEM). For example, refer to Refs. [2-6]. In the previous works, the shape functions were defined in the global coordinate system and seeking the numerical solution of the Laplace equation was unavoidable. For instance, the methods such as finite difference method [2], the method of fundamental solutions [4], the boundary element method [5], the complex variable boundary method [6] and the FEM is used [3] to approximate the harmonic shape functions

To avoid the numerical solution of the Laplace equation, the shape functions are derived here in the local coordinate system for a representative master element. Therefore, the harmonic shape functions can be obtained analytically using the Fourier series. In addition, a new procedure is proposed here for systematic computation of the shape functions. In this approach, a small set of harmonic functions are defined at first as the building blocks and then all of the shape functions of different elements with different node arrangements are derived using proper transformations of them.

To evaluate the performance and the potentials of the proposed method the Laplace equation is selected in the present manuscript as the model equation. Two benchmark examples are solved and the results are presented. In these examples, the patch test and convergence analysis are done. The results have shown that the proposed method treats non-conforming elements very naturally and efficiently. A lot of development is still to be done in the applications of harmonic shape functions and we believe it can begin an interesting field of research.

2- QUADRILATERAL ELEMENTS WITH MIDSIDE NODES

Consider a set of quadrilateral elements with midside nodes and linear variation of the shape functions on the element boundaries. In the most general case, the quadrilateral elements with midside nodes consist of 5 different element types which are shown schematically in Fig. 1. All of these elements are compatible with traditional linear Lagrange elements and a composite mesh with standard elements and elements with harmonic shape functions is possible.

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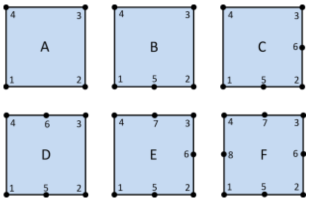

Fig. 1. Different elements with midside nodes

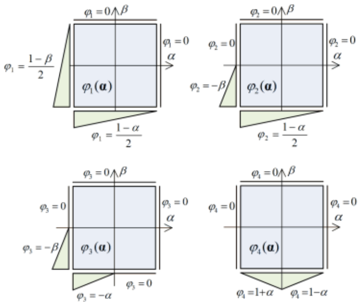

Fig. 2. Boundary conditions for obtaining RFs

3- ROOT FUNCTIONS

Different element types in Fig. 1 contain a total of 32 nodes and each node has its own shape function. All of these shape functions can be constructed using only 4 Root Functions (RFs) and proper coordinate transformations. To explain this more consider Fig. 2. In this Figure, the boundary conditions for obtaining the RFs on the boundaries of the square domain α, β [-1,1] in the Root Coordinate System (RCS) are given.

The value of the RFs on the entire boundary of the square domain is given in Fig. 2. In fact, we seek the value of the RFs at the internal points of the square domain. In the present work, the Laplace boundary value problem is used here to obtain RFs in the internal points. The method of separation of variables and series solution are used to obtain the analytic solution of the Laplace boundary value problem [7]. The general solution of this boundary value problem is as follows.

$$\varphi=\sum_{n=1}^{\infty}\left(A_1^n\ A_2^n(\alpha)\ A_3^n(\beta)+B_1^n\ B_2^n(\alpha)\ B_3^n(\beta)\right) \tag{1}$$

where the coefficients A_1^n to B_3^n are dependent on boundary conditions given in Fig. 2. The surface plots of the four root functions are shown in Fig. 3.

4- HARMONIC SHAPE FUNCTIONS

The shape functions of element types A to E can be obtained using the four RFs with proper coordinate transformations. The general formula for the shape functions is given here as follows.

$$N(\boldsymbol{\xi})=\varphi(\boldsymbol{T}\boldsymbol{\xi}) \tag{2}$$

were N is the harmonic shape function defined in Local Coordinate System (LCS), φ is a root function in RCS and T is a transformation matrix that transforms the local coordinates $\boldsymbol{\xi}$ to the root coordinates $\boldsymbol{\alpha}$. For example, the surface plots of the shape functions of element type B are shown in Fig. 4.

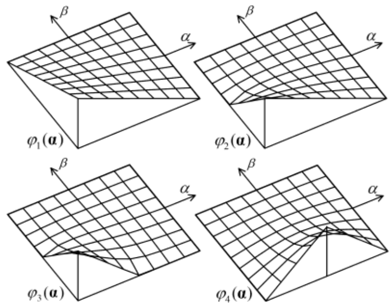

Fig. 3. Four root functions

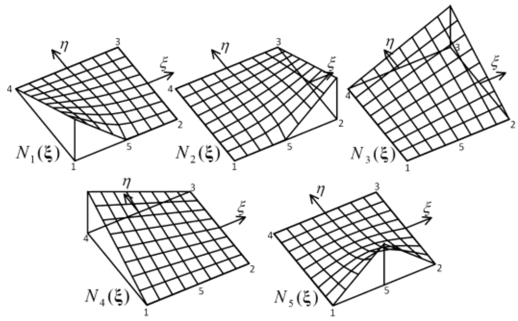

Fig. 4. The shape functions of element type B

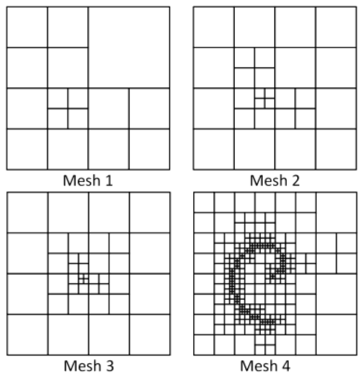

Fig. 5. The meshes of the first example

5- NUMERICAL EXAMPLES

As the first example, the patch test is conducted for the Laplace equation over a unit square using different meshes which are shown in Fig. 5. The exact solution is considered as $T=x+3y$ and the Dirichlet boundary condition is applied on all of the boundaries of the square domain. The Laplace equation was solved using the proposed method and the results showed that the proposed method would pass the patch test.

In the second example, a convergence test for the Laplace equation is done to evaluate the effect of mesh size on the quality of the results. To do this, assume the Laplace equation holds in a unit square domain and consider the exact solution as $u=e^x\sin(y)$. Dirichlet boundary condition is applied to the domain boundaries using this exact solution. The problem is solved using four different meshes as shown in Fig. 6. The relative L_2 error norm of the field variable for each mesh is presented in Fig. 7.

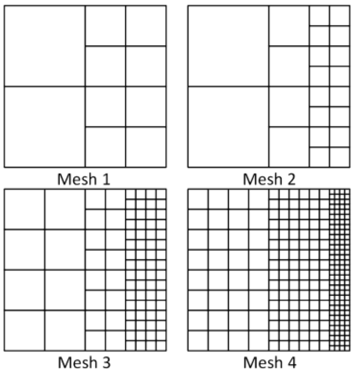

Fig. 6. The meshes of the second example

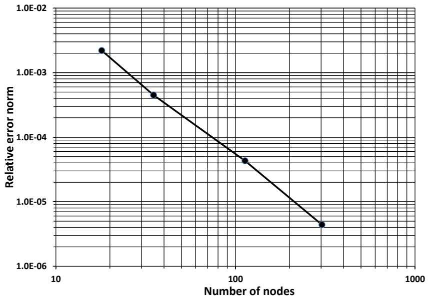

Fig. 7. Relative error norm in the second example

6- CONCLUSIONS

The harmonic functions were used in the present work to define the elements with midside nodes. The shape functions were defined in such a way that vary linearly between adjacent nodes on the element boundaries and therefore the elements could be attached to any standard linear elements. As a result, the hanging nodes were treated very naturally in the present method without any additional constraints. Two numerical examples were solved to evaluate the proposed method and the results are presented.

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HOW TO CITE THIS ARTICLE

M. J. Kazemzadeh-Parsi, Application of Nonconforming Quadtree Grids in the Finite Element Method, Amirkabir J. Mech Eng., 53(2) (2021) 167-170.

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